

SPECTRUM OF THE SEMI-RELATIVISTIC PAULI-FIERZ MODEL II

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Abstract

The existence of the ground state of the so-called semi-relativistic Pauli-Fierz model is proven. Let A be a quantized radiation field and $H_{f,m}$ the free field Hamiltonians which is the second quantization of $\sqrt{|k|^2 + m^2}$. It is established in [HH13a] that the *massive* ($m > 0$) semi-relativistic Pauli-Fierz model

$$\sqrt{(-i\nabla \otimes \mathbb{1} - A)^2 + M^2} + V \otimes \mathbb{1} + \mathbb{1} \otimes H_{f,m}$$

in quantum electrodynamics has the unique ground state. I.e., $(m, M) \in (0, \infty) \times [0, \infty)$ is assumed. In this paper the existence of the ground state for the *massless* ($m = 0$) semi-relativistic Pauli-Fierz model is shown. I.e., $(m, M) \in [0, \infty) \times [0, \infty)$ is assumed. We emphasize that our results include a singular case of $(m, M) = (0, 0)$, i.e., the Hamiltonian is of the form:

$$|-i\nabla \otimes \mathbb{1} - A| + V \otimes \mathbb{1} + \mathbb{1} \otimes H_f,$$

where H_f is the second quantization of $|k|$.

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1 Preliminaries

1.1 Introduction

In this paper we are concerned with the existence of the ground state of the so-called semi-relativistic Pauli-Fierz (it is shorthand as "SRPF" in this paper) model in quantum electrodynamics. This model describes a minimal interaction between a semi-relativistic quantum matter and a quantized radiation field $A = (A_1, A_2, A_3)$. The matter is governed by the semi-relativistic Schrödinger operator defined by $\sqrt{-\Delta + M^2} + V$, where M denotes the mass of the matter and V an external potential. On the other hand the free field Hamiltonian is given by $H_{f,m}$ where $H_{f,m}$ is the second quantization of $\omega(k) = \sqrt{|k|^2 + m^2}$ and m describes the mass of a boson. Then the decoupled Hamiltonian is defined by

$$\left(\sqrt{-\Delta + M^2} + V\right) \otimes \mathbb{1} + \mathbb{1} \otimes H_{f,m}$$

in a product Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathcal{F}$, where \mathcal{F} denotes a boson Fock space. The minimal coupling implies to replace $-\Delta \otimes \mathbb{1}$ with $(p \otimes \mathbb{1} - A)^2$, where $p_\mu = -i\nabla_\mu$ with the generalized differential operator ∇_μ , and

$$A_\mu = \int_{\mathbb{R}^3}^{\oplus} A_\mu(x) dx, \quad \mu = 1, 2, 3.$$

Thus the Hamiltonian of SRPF is given by

$$H_m = \sqrt{(p \otimes \mathbb{1} - A)^2 + M^2} + V \otimes \mathbb{1} + \mathbb{1} \otimes H_{f,m}.$$

Note that the boson-mass m should be however zero since the boson physically describes a photon. We show that H_m is self-adjoint in [HH13b] and the spectrum of H_m is

$$\sigma(H_m) = \{E_m\} \cup [E_m + m, \infty), \quad (1.1)$$

where $E_m = \inf \sigma(H_m)$ is the bottom of the spectrum of H_m . Eigenvector associated with E_m is called a ground state of H_m . It is suggested to study the ground state of SRPF Hamiltonian in [GLL01] where the existence of the ground state of the Pauli-Fierz Hamiltonian

$$\frac{1}{2M}(p \otimes \mathbb{1} - A)^2 + V \otimes \mathbb{1} + \mathbb{1} \otimes H_{f,0}$$

is proven under the binding condition. Then SRPF Hamiltonian has been studied so far in e.g. [GS12, HH13b, Hir14, HS10, KM13a, KM13b, KMS11a, KMS11b, MS10, MS09, O16, S13a, S13b]. In particular when $m = 0$ but $M > 0$, one can also show the existence of ground state. In this case the bottom of the spectrum is the edge of the continuous spectrum and it requires a non-perturbative analysis. This is actually done in [KMS11a, KMS11b]. Results related to binding condition and enhanced binding are also given in [GS12, KM13b, S13a, S13b]. It is also shown that H_m has a ground state for $m > 0$ but $M = 0$ in [HH13b], where the ground state energy is discrete but the assumption $M = 0$ produces a singularity. It is emphasized again that E_m for $m = 0$ is the edge of the continuous spectrum and there is no positive gap between E_m and $\inf \sigma(H_m) \setminus \{E_m\}$. Furthermore E_m is simple which is shown in [Hir14, Corollary 6.2].

Then a remaining problem is to study the case of $(m, M) = (0, 0)$, i.e.,

$$H_0 = |p \otimes \mathbb{1} - A| + V \otimes \mathbb{1} + \mathbb{1} \otimes H_{f,0}, \quad (1.2)$$

and we solve this in this paper, i.e., the existence of the ground state of (1.2) is shown.

1.2 Applications to asymptotic field and outline of proofs

Let $M = 0$. In order to avoid the infrared divergence we unitarily transform H_m to a regularized Hamiltonian H_m^R , which is of the form

$$H_m^R = |p \otimes \mathbb{1} - A_R| + V \otimes \mathbb{1} + \mathbb{1} \otimes H_{f,m} + h.$$

Here A_R is given (2.9) and h in (2.8) below. Note that H_m is unitarily equivalent to H_m^R :

$$H_m \cong H_m^R.$$

This transformation is initially used in [BFS99]. In this paper the asymptotic annihilation operator defined by

$$a_{\pm\infty}(f, j) = \text{s-} \lim_{t \rightarrow \pm\infty} e^{-itH_m^R} e^{itH_{f,m}} a(f, j) e^{-itH_{f,m}} e^{itH_m^R}$$

is applied to prove the existence of ground state. This is established in [AHH99, Hir05] and reviewed in Appendix A. Let $\langle x \rangle^2 = \sqrt{|x|^2 + 1}$ as usual. In order to show the existence of the ground state it is enough to check three uniform bounds concerning Φ_m for $m > 0$:

(A) spatial decay: $\sup_{x \in \mathbb{R}^3} \|\langle x \rangle^2 \Phi_m(x)\|_{\mathcal{F}} < C,$

(B) the number of bosons: $\|N^{\frac{1}{2}}\Phi_m\|_{\mathcal{H}} < C$, where N denotes the number operator,

(C) Sobolev norm of n -particle sector: $\sup_{0 < m \leq m_0} \|\Phi_m^{(n)}\|_{W^{1,p}(\Omega)}$ for $1 \leq p < 2$ and any finite domain $\Omega \subset \mathbb{R}_x^3 \times \mathbb{R}_k^{3n}$.

We review (A),(B) and (C) below:

(A) The spatial exponential decay

$$\sup_{x \in \mathbb{R}^3} \|e^{|x|}\Phi_m(x)\|_{\mathcal{F}} \leq C \quad (1.3)$$

with some C independent of m is fortunately established in [Hir14, Theorem 5.12]. See also [GLL01, BFS99]. This implies $\sup_{x \in \mathbb{R}^3} \|\langle x \rangle^2 \Phi_m(x)\|_{\mathcal{F}} < C'$.

(B) The number of bosons in Φ_m can be uniformly estimated in m , i.e.,

$$\|N^{\frac{1}{2}}\Phi_m\| < C, \quad (1.4)$$

where C is independent of m . To derive this inequality we use the identity:

$$a(f, j)\Phi_m = - \int_{\mathbb{R}^3} f(k)(H_m^R - E_m + \omega(k))^{-1} C_j(k) \langle x \rangle^2 \Phi_m dk, \quad (1.5)$$

where $C_j(k)$ denotes a bounded operator for each $k \in \mathbb{R}^3$. See Lemma 3.13 for the explicit statement. (1.5) can be derived by the Cook method in scattering theory and the fact

$$a_{\pm\infty}(f, j)\Phi_m = 0. \quad (1.6)$$

It can be also established that the map

$$T_{gj} : L^2(\mathbb{R}^3) \ni f \mapsto - \int_{\mathbb{R}^3} f(k)(H_m^R - E_m + \omega(k))^{-1} C_j(k) \langle x \rangle^2 \Phi_m dk \in \mathcal{H}$$

is a Hilbert-Schmidt operator if and only if $\Phi_m \in D(N^{\frac{1}{2}})$, and if $\Phi_m \in D(N^{\frac{1}{2}})$, then

$$\sum_{j=1,2} \|T_{gj}\|_{\text{HS}}^2 = \sum_{j=1,2} \int_{\mathbb{R}^3} \|(H_m^R - E_m + \omega(k))^{-1} C_j(k) \langle x \rangle^2 \Phi_m\|^2 dk = \|N^{\frac{1}{2}}\Phi_m\|^2. \quad (1.7)$$

See Proposition 3.8. It is proven in several literatures that this type of argument is very useful to show the existence of the ground state. In order to derive (1.5) we have to compute the commutator:

$$[H_m^R, a(f, j)]\Phi_m = [|p \otimes \mathbb{1} - A_R|, a(f, j)]\Phi_m + [\mathbb{1} \otimes H_{f,m}, a(f, j)]\Phi_m + [h, a(f, j)]\Phi_m$$

for $m > 0$. Since $|p \otimes \mathbb{1} - A|$ is a non-local operator and not smooth by missing positive mass term M , it is crucial to see (1) and (2) below:

(1) to find a dense domain \mathcal{D} such that $\mathcal{D} \subset D(|p \otimes \mathbb{1} - A_R| a(f, j)) \cap D(a(f, j) |p \otimes \mathbb{1} - A_R|)$,

(2) to show the boundedness of $C_j(k)$.

(1) is needed to guarantee that $[p \otimes \mathbb{1} - A, a(f, j)]\Phi_m$ is well-defined, and (2) is used to see (1.7). In this paper we show (1) and (2) in Lemma 3.6 and Lemma 3.9, respectively. We emphasize that it is not trivial to show both of them. Consequently by virtue of (1) and (2) it can be derived that $\|N^{\frac{1}{2}}\Phi_m\| \leq C\|\langle x \rangle^2\Phi_m\|$ and the spatial exponential decay (1.3) yields (1.4).

(C) Let \mathcal{H} be decomposed into n -particle sectors: $\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}^{(n)}$. It is shown that n -particle sector of Φ_m satisfies that $\Phi_m^{(n)} \in W^{1,p}(\Omega)$ for any bounded $\Omega \subset \mathbb{R}_x^3 \times \mathbb{R}_k^{3n}$ and

$$\sup_{0 < m < m_0} \|\Phi_m^{(n)}\|_{W^{1,p}(\Omega)} < \infty, \quad n \geq 1. \quad (1.8)$$

We derive this in a different way from [GLL01], where this method is initiated. (1.8) can be also shown by using (1.5) as follows: Let $\Psi = (0, \dots, 0, \overset{n_{th}}{G}, 0, \dots)$ with $G \in \mathcal{H}^{(n)}$. Noting that

$$(\Psi, a(f, j)\Phi_m)_{\mathcal{H}^{(n)}} = (a^\dagger(\bar{f}, j)\Psi, \Phi_m)_{\mathcal{H}^{(n+1)}} = \sqrt{n+1}(\bar{f} \otimes \Psi, \Phi_m)_{\mathcal{H}^{(n+1)}},$$

we take the inner product of both sides of (1.5)

$$(\Psi, a(f, j)\Phi_m)_{\mathcal{H}^{(n)}} = -(\bar{f} \otimes G, (H_m^R - E_m + \omega(\cdot))^{-1}C_j(\cdot)\langle x \rangle^2\Phi_m)_{\mathcal{H}^{(n+1)}}.$$

Thus we have the identity

$$(\nabla_\mu f \otimes G, \Phi_m)_{\mathcal{H}^{(n+1)}} = \frac{1}{\sqrt{n+1}}(f \otimes G, \nabla_\mu (H_m^R - E_m + \omega(\cdot))^{-1}C_j(\cdot)\langle x \rangle^2\Phi_m)_{\mathcal{H}^{(n+1)}}$$

by the integral by parts formula. Hence the right-hand side can be estimated and conclude (1.8) in Lemmas 3.30 and 3.31.

Finally combining (A),(B) and (C) we can show that the normalized ground state Φ_m strongly converges to a nonzero vector as $m \rightarrow 0$, which is nothing but the ground state of H_m^R , i.e., H_m , for $m = 0$.

This paper organized as follows. In Section 2 we give the definition of SRPF Hamiltonian H_m as a self-adjoint operator, and introduce a regularized SRPF Hamiltonian H_m^R . Section 3 is devoted to proving $\|N^{\frac{1}{2}}\Phi_m\| < C$ and $\sup_{0 < m < m_0} \|\Phi_m^{(n)}\|_{W^{1,p}(\Omega)} < \infty$, and then show the main theorem in Theorem 3.33. In Appendix we review asymptotic field used in this paper.

2 Semi-relativistic Pauli-Fierz Hamiltonian

2.1 Definition of semi-relativistic Pauli-Fierz model

We define the Hamiltonian of SRPF model as a self-adjoint operator on a Hilbert space. As is mentioned in the previous section the Hamiltonian of SRPF model includes non-local operator, hence the definition of the self-adjoint operator is not straightforward. The

operator consists of a matter part and quantum field part. We firstly introduce the quantum field part.

Let us introduce the boson Fock space. The boson Fock space, \mathcal{F} , over Hilbert space $W = L^2(\mathbb{R}^3 \times \{1, 2\})$ is given by

$$\mathcal{F} = \oplus_{n=0}^{\infty} \mathcal{F}_n(W) = \oplus_{n=0}^{\infty} [\otimes_s^n W],$$

where $\otimes_s^n W$ denotes the symmetric tensor product of W and $\otimes_s^0 W = \mathbb{C}$. On \mathcal{F} the scalar product is defined by $(\Phi, \Psi) = \sum_{n=0}^{\infty} (\Phi^{(n)}, \Psi^{(n)})_{\otimes^n W}$. Then $\Psi \in \mathcal{F}$ can be identified a sequence $\{\Psi^{(n)}\}_{n=0}^{\infty}$ such that $\sum_{n=0}^{\infty} \|\Psi^{(n)}\|^2 < \infty$. In particular the Fock vacuum is given by $\Omega = (1, 0, 0, \dots) \in \mathcal{F}$.

Let T be a densely defined closable T in W . The second quantization of T is the closed operator in \mathcal{F} , which is defined by

$$d\Gamma(T) = \oplus_{n=0}^{\infty} (T^{(n)}),$$

where $\otimes^0 T = \mathbb{1}$, $T^{(n)} = \sum_{k=1}^n \mathbb{1} \otimes \dots \otimes \mathbb{1} \otimes T \otimes \mathbb{1} \otimes \dots \otimes \mathbb{1}$ with $T^{(0)} = 0$. If T is a non-negative self-adjoint operator in W , then $d\Gamma(T)$ turns to be also a non-negative self-adjoint operator. We denote the spectrum (resp. point spectrum) of T by $\sigma(T)$ (resp. $\sigma_P(T)$). The Fock vacuum Ω the eigenvector of $d\Gamma(T)$ associated with eigenvalue 0, i.e., $d\Gamma(T)\Omega = 0$. The number operator, N , is defined by the second quantization of the identity $\mathbb{1}$ on W :

$$N = d\Gamma(\mathbb{1}),$$

and $\sigma(N) = \mathbb{N} \cup \{0\}$. Let $\omega(k) = \sqrt{|k|^2 + m^2}$, $k \in \mathbb{R}^3$, be a dispersion relation and it can be regarded as the multiplication operator in W . The free field Hamiltonian $H_{f,m}$ is given by the second quantization of ω :

$$H_{f,m} = d\Gamma(\omega).$$

Then $H_{f,m}$ is a non-negative self-adjoint operator in \mathcal{F} , and we see that

$$\sigma(H_{f,m}) = \{0\} \cup [m, \infty), \quad \sigma_P(H_{f,m}) = \{0\}. \quad (2.1)$$

Moreover $H_{f,m}\Omega = 0$. The creation operator $a^\dagger(f)$ smeared by $f \in W$ is given by

$$(a^\dagger(f)\Psi)^{(n)} = \sqrt{n} S_n(f \otimes \Psi^{(n-1)}), \quad n \geq 1,$$

and $(a^\dagger(f)\Psi)^{(0)} = 0$ with the domain:

$$D(a^\dagger(f)) = \left\{ \Psi \in \mathcal{F}_b \mid \sum_{n=1}^{\infty} \|\sqrt{n} S_n(f \otimes \Psi^{(n-1)})\|_{\otimes^n W}^2 < \infty \right\}.$$

Here S_n is the symmetrization operator on $\otimes^n W$. The annihilation operator smeared by $f \in W$ is given by the adjoint of $a^\dagger(\bar{f})$: $a(f) = (a^\dagger(\bar{f}))^*$. Both $a(f)$ and $a^\dagger(f)$ are linear in f , and satisfy canonical commutation relations:

$$[a(f), a^\dagger(g)] = (\bar{f}, g)_W, \quad [a(f), a(g)] = 0 = [a^\dagger(f), a^\dagger(g)].$$

We formally write $a^\sharp(f) = \sum_{j=1,2} \int a^\sharp(k, j) f(k, j) dk$ for $a^\sharp(f)$. Let us introduce the finite particle subspace \mathcal{F}_{fin} by

$$\mathcal{F}_{\text{fin}} = \text{L.H.}\{\Omega, a^\dagger(h_1) \cdots a^\dagger(h_n) \Omega | h_j \in C_0^\infty(\mathbb{R}^3) \oplus C_0^\infty(\mathbb{R}^3), j = 1, \dots, n, n \geq 1\},$$

which is a dense subspace of \mathcal{F} . We shall define a quantized radiation field $A(x)$. Let $e(\cdot, 1)$ and $e(\cdot, 2)$ be polarization vectors i.e., $e(k, j) \cdot e(k, j') = \delta_{jj'}$ and $k \cdot e(k, j) = 0$ for $k \in \mathbb{R}^3 \setminus \{0\}$ and $j, j' = 1, 2$, and we choose

$$e(k, 1) = \frac{(k_2, -k_1, 0)}{\sqrt{k_1^2 + k_2^2}}, \quad e(k, 2) = \frac{k}{|k|} \times e(k, 1). \quad (2.2)$$

Note that $e(\cdot, j) \in C^\infty(\mathbb{R}^3 \setminus \{0\})$ for $j = 1, 2$. For each $x \in \mathbb{R}^3$ the quantized radiation field, $A(x) = (A_1(x), A_2(x), A_3(x))$, is given by

$$A_\mu(x) = \frac{1}{\sqrt{2}} \sum_{j=1,2} \int e_\mu(k, j) (a^\dagger(k, j) \phi_\omega(k) e^{-ikx} + a(k, j) \phi_\omega(-k) e^{ikx}) dk, \quad (2.3)$$

where

$$\phi_\omega = \frac{\hat{\varphi}}{\sqrt{\omega}}$$

and $\hat{\varphi}$ is a cutoff function. In addition the conjugate momentum is as usual defined by

$$\Pi_\mu(x) = \frac{i}{\sqrt{2}} \sum_{j=1,2} \int e_\mu(k, j) (a^\dagger(k, j) \phi_\omega(k) e^{-ikx} - a(k, j) \phi_\omega(-k) e^{ikx}) dk. \quad (2.4)$$

Note that

$$i[N, A_\mu(x)] = \Pi_\mu(x).$$

If $\hat{\varphi}(k) = \overline{\hat{\varphi}(-k)}$ and $\hat{\varphi}/\sqrt{\omega} \in L^2(\mathbb{R}^3)$, by Nelson's analytic vector theorem, for each $x \in \mathbb{R}^3$ $A_\mu(x)$ and $\Pi_\mu(x)$ are essentially self-adjoint.

Then let us introduce assumptions on ultraviolet cutoff function $\hat{\varphi}$.

Assumption 2.1 *Ultraviolet-cutoff function $\hat{\varphi}$ satisfies that (1) $\hat{\varphi} \in C_0^\infty(\mathbb{R}^3)$ and (2) $\hat{\varphi}(k) = \overline{\hat{\varphi}(-k)}$.*

Statement (2) of Assumption 2.1 implies that $\omega^n \phi_\omega \in L^2(\mathbb{R}^3)$ for any $n \in \mathbb{N}$, which yields together with (1) that SRPF Hamiltonian is self-adjoint and (2) is also used to establish a derivative bound of the massive ground state, which is studied in Section 3.3. Let $\overline{A_\mu(x)}$ be

the closure of $A_\mu(x)$, and then it is self-adjoint. We define the self-adjoint operator A_μ by $\int_{\mathbb{R}^3}^\oplus \overline{A_\mu(x)} dx$ and we set $A = (A_1, A_2, A_3)$.

We shall explain the particle part. Let $p = (p_1, p_2, p_3) = (-i\nabla_1, -i\nabla_2, -i\nabla_3)$ be the momentum operator of particle. Then the particle Hamiltonian under consideration is a relativistic Schrödinger operator given by

$$H_p = \sqrt{-\Delta + M^2} + V$$

in $L^2(\mathbb{R}^3)$. Here $V : \mathbb{R}^3 \rightarrow \mathbb{R}$ denotes an external potential.

Finally we define the total Hamiltonian of SRPF model, which is an operator in the Hilbert space

$$\mathcal{H} = L^2(\mathbb{R}_x^3) \otimes \mathcal{F} = \oplus_{n=0}^\infty \mathcal{H}^{(n)} = \oplus_{n=0}^\infty L^2(\mathbb{R}_x^3) \otimes \mathcal{F}_n(W)$$

and is given by the minimal coupling of the decoupled Hamiltonian

$$H_p \otimes \mathbb{1} + \mathbb{1} \otimes H_{f,m} \tag{2.5}$$

by quantized radiation field A , i.e., $p_\mu \otimes \mathbb{1}$ is replaced by $p_\mu \otimes \mathbb{1} - A$ in (2.5). Let $C^\infty(T) = \cap_{n=1}^\infty D(T^n)$.

Proposition 2.2 *Suppose Assumption 2.1. Then $(p \otimes \mathbb{1} - A)^2$ is essentially self-adjoint on $D(-\Delta) \cap C^\infty(N)$.*

Proof: See [Hir14, Proposition 3.4]. ■

The closure of $(p \otimes \mathbb{1} - A)^2 \upharpoonright_{D(-\Delta) \cap C^\infty(N)}$ is denoted by $(p \otimes \mathbb{1} - A)^2$ in what follows. Thus $\sqrt{(p \otimes \mathbb{1} - A)^2 + M^2 \otimes \mathbb{1}}$ is defined through the spectral measure of $(p \otimes \mathbb{1} - A)^2$. We shall give a firm definition of SRPF-Hamiltonian below.

Definition 2.3 *Let $(m, M) \in [0, \infty) \times [0, \infty)$. Suppose Assumption 2.1. Then SRPF Hamiltonian is defined by*

$$H_m = \sqrt{(p \otimes \mathbb{1} - A)^2 + M^2 \otimes \mathbb{1}} + V \otimes \mathbb{1} + \mathbb{1} \otimes H_{f,m}$$

with the domain

$$D(H_m) = D(\sqrt{(p \otimes \mathbb{1} - A)^2 + M^2 \otimes \mathbb{1}}) \cap D(V \otimes \mathbb{1}) \cap D(\mathbb{1} \otimes H_{f,m}).$$

We do not write tensor notation \otimes for notational convenience in what follows. Thus H_m can be simply written as

$$H_m = \sqrt{(p - A)^2 + M^2} + V + H_{f,m}.$$

Let $\mathcal{H}_{\text{fin}} = C_0^\infty(\mathbb{R}^3) \hat{\otimes} \mathcal{F}_{\text{fin}}$, where $\hat{\otimes}$ denotes the algebraic tensor product. Let us introduce classes of external potentials studied in this paper.

Definition 2.4 (External potentials)

(V_{rel}) $V \in V_{\text{rel}}$ if and only if $D(\sqrt{-\Delta + M^2}) \subset D(V)$ and there exist $0 \leq a < 1$ and $b \geq 0$ such that for all $f \in D(\sqrt{-\Delta + M^2})$,

$$\|Vf\| \leq a\|\sqrt{-\Delta + M^2}f\| + b\|f\|.$$

(V_{conf}) $V = V_+ - V_- \in V_{\text{conf}}$ if and only if $V_- = 0$ and $V = V_+$ satisfies that V is twice differentiable, and $\nabla_\mu V, \nabla_\mu^2 V \in L^\infty(\mathbb{R}^3)$ for $\mu = 1, 2, 3$, and $D(V) \subset D(|x|)$.

Examples of V_{rel} and V_{conf} include that $V(x) = -Z/|x| \in V_{\text{rel}}$, and $V(x) = \langle x \rangle \in V_{\text{conf}}$.

Proposition 2.5 ([HH13a, Theorem 1.9]) *Let $(m, M) \in [0, \infty) \times [0, \infty)$. Suppose that Assumption 2.1 holds and $V \in V_{\text{conf}} \cup V_{\text{rel}}$. Then H_m is self-adjoint on $D(|p|) \cap D(V) \cap D(H_{f,m})$, and essentially self-adjoint on \mathcal{H}_{fin} .*

Let T be a self-adjoint operator in a Hilbert space and $E = \inf \sigma(T)$. The eigenvector f such that $Tf = Ef$ is called the ground state of T . Note that ground states do not necessarily exist. When $(m, M) \in (0, \infty) \times [0, \infty)$ however the existence of the ground state of H_m is established in e.g., [HH13b]. The main purpose of this paper is to show the existence of the ground state of H_m for any $(m, M) \in [0, \infty) \times [0, \infty)$. In particular it is emphasized that the existence of the ground state for the case of $(m, M) = (0, 0)$ has not been shown so far as far as we know. There are however several results concerning the ground state for $(m, M) \in (0, \infty) \times [0, \infty)$. Then in this paper we assume the existence of ground state of H_m for $m > 0$. Then we introduce several assumptions on the ground state φ_m of H_m .

Assumption 2.6 (1) H_m has the ground state φ_m for all $(m, M) \in (0, \infty) \times [0, \infty)$.

(2) There exists $m_0 > 0$ such that

$$\sup_{0 < m < m_0} \|\langle x \rangle^2 \varphi_m\| < \infty. \quad (2.6)$$

Let us denote the essential spectrum of H_m by $\sigma_{\text{ess}}(H_m)$.

Proposition 2.7 (Exponential decay of φ_m) *Let $V \in V_{\text{conf}}$ and we suppose that $(m, M) \in (0, \infty) \times [0, \infty)$. Suppose Assumption 2.1. Then (1) and (2) follow:*

(1) $\sigma_{\text{ess}}(H_m) = [E_m + m, \infty)$. In particular H_m has a ground state φ_m .

(2) For all $x \in \mathbb{R}^3$, $\|\varphi_m(x)\|_{\mathcal{F}} \leq Ce^{-c|x|}$ with some constants $c > 0$ and $C > 0$ independent of $(m, M) \in (0, \infty) \times [0, \infty)$.

Proof: See [HH13a, Theorem 2.8] for the proof of statement (1), and [HH13b, Corollary 2.9] and [Hir14, Theorem 5.12] for (2). ■

Proposition 2.7 gives an example of φ_m such that conditions in Assumption 2.6 are satisfied.

2.2 Regularized SRPF Hamiltonians

We transform H_m to a certain regular Hamiltonian to avoid the infrared divergence. Let us define the unitary operator $U = \int_{\mathbb{R}^3}^{\oplus} U(x) dx$ on \mathcal{H} by

$$U(x) = \exp(ix \cdot A(0)),$$

and we set

$$H_{f,m}^R = H_{f,m} + \int_{\mathbb{R}^3}^{\oplus} h(x) dx, \quad (2.7)$$

$$h(x) = -i \sum_{j=1,2} \int x \cdot e(k, j) \phi_{\omega}(k) (a^{\dagger}(k, j) - a(k, j)) dk + \|\hat{\varphi}e(\cdot, j) \cdot x\|^2. \quad (2.8)$$

Here $A(0)$ is defined by $A(x)$ with x replaced by 0. We simply write h for $\int_{\mathbb{R}^3}^{\oplus} h(x) dx$. Formally (2.7) is represented as

$$H_{f,m}^R = \sum_{j=1,2} \int_{\mathbb{R}^3}^{\oplus} \left(\int \omega(k) b_j^{\dagger}(k, x) b_j(k, x) dk \right) dx,$$

where $b_j(k, x) = a(k, j) - i\phi_{\omega}(k)e(k, j) \cdot x$ for each $x \in \mathbb{R}^3$. Let

$$A_{R\mu}(x) = A_{\mu}(x) - A_{\mu}(0) \quad (2.9)$$

and $A_R = \int_{\mathbb{R}^3}^{\oplus} A_R(x) dx$. Thus

$$A_{R\mu}(x) = \frac{1}{\sqrt{2}} \sum_{j=1,2} \int e_{\mu}(k, j) (a^{\dagger}(k, j) \phi_{\omega}(k) (e^{-ikx} - 1) + a(k, j) \phi_{\omega}(-k) (e^{ikx} - 1)) dk.$$

In a similar manner to Proposition 2.2, we can also see that $(p - A_R)^2$ is essentially self-adjoint on $D(-\Delta) \cap C^{\infty}(N)$, and the closure of $(p - A_R)^2|_{D(-\Delta) \cap C^{\infty}(N)}$ is denoted by $(p - A_R)^2$. Let

$$H_m^R = \sqrt{(p - A_R)^2 + M^2} + H_{f,m} + h + V.$$

Proposition 2.8 *Let $(m, M) \in [0, \infty) \times [0, \infty)$. Suppose Assumptions 2.1. Then H_m^R is self-adjoint on $U^{-1}D(H_m)$ and essentially self-adjoint on $U^{-1}\mathcal{H}_{\text{fin}}$, and it follows that*

$$H_m^R = U^{-1}H_mU \quad (2.10)$$

on $U^{-1}D(H_m)$

Proof: We can see that $U^{-1}(p - A)^2U = (p - A_R)^2$ on $D(\Delta) \cap C^{\infty}(N)$, and $D(\Delta) \cap C^{\infty}(N)$ is a core of both $(p - A)^2$ and $(p - A_R)^2$. Hence U maps $D((p - A_R)^2)$ to $D((p - A)^2)$ and $U^{-1}(p - A)^2U = (p - A_R)^2$ holds as self-adjoint operators, and

$$U^{-1}\sqrt{(p - A)^2 + M^2}U = \sqrt{(p - A_R)^2 + M^2}$$

also holds true. In particular $U^{-1}\sqrt{(p - A)^2 + M^2}U = \sqrt{(p - A_R)^2 + M^2}$ holds on \mathcal{H}_{fin} . Furthermore $U^{-1}H_{f,m}U = H_{f,m}^R$ on \mathcal{H}_{fin} . Then (2.10) holds on \mathcal{H}_{fin} . Since \mathcal{H}_{fin} is a core of H_m , (2.10) follows from a limiting argument. Furthermore H_m^R is essentially self-adjoint on $U^{-1}\mathcal{H}_{\text{fin}}$ and self-adjoint on $U^{-1}D(H_m)$. \blacksquare

2.3 Infrared singularity

In what follows we study H_m^{R} instead of H_m . An advantage of studying H_m^{R} is that we do *not* need the infrared regular condition:

$$\int_{\mathbb{R}^3} \frac{|\hat{\varphi}(k)|^2}{\omega(k)^3} dk < \infty \quad (2.11)$$

to show the existence of the ground state. Physically reasonable choice of $\hat{\varphi}(0)$ is nonzero, since $\hat{\varphi}(0)$ amounts to the charge. Then by the singularity at the origin, $\int_{\mathbb{R}^3} \frac{|\hat{\varphi}(k)|^2}{\omega(k)^3} dk = \infty$ when $m = 0$. Actually instead of (2.11) we need the condition:

$$\int_{\mathbb{R}^3} (|k| + |k|^2)^2 \frac{\hat{\varphi}(k)^2}{\omega(k)^3} dk < \infty$$

to show the existence of ground state. Thus in the case of $m = 0$ and $\hat{\varphi}(0) \neq 0$ we can also show the existence of the ground state. See Theorem 3.33 and Corollary 3.14.

3 Infrared bounds

Throughout we assume Assumption 2.1. Let Φ_m be a normalized ground state of H_m^{R} . Note that $\sup_{0 < m < m_0} \|\langle x \rangle^2 \Phi_m\| < \infty$ since $U\Phi_m$ is a ground state of H_m^{R} , and $[\langle x \rangle^2, U] = 0$. In this section we shall prove two bounds concerning Φ_m by using the so-called pull-through formula. In this section we set

$$M = 0.$$

Then

$$H_m^{\text{R}} = |p - A_{\text{R}}| + V + H_{\text{f},m} + h.$$

3.1 Stability of a domain

For notational simplicity we set

$$T_p = (p - A_{\text{R}})^2.$$

Then

$$\sqrt{T_p} = |p - A_{\text{R}}|.$$

Let

$$H_{\text{int}} = H_m^{\text{R}} - (|p| + H_{\text{f},m}) = \sqrt{T_p} - |p| + h.$$

Then $H_m^{\text{R}} = |p| + H_{\text{f}} + H_{\text{int}}$. The pull-through formula we see later is a useful tool to study the ground state associated with embedded eigenvalues. In order to establish the pull-through formula we begin with establishing that $[\sqrt{T_p}, a(f, j)]$ is well defined on some *dense* domain \mathcal{D} , i.e.,

$$\text{D}(a(f, j)\sqrt{T_p}) \cap \text{D}(\sqrt{T_p}a(f, j)) \supset \mathcal{D}.$$

In order to find \mathcal{D} we apply a stochastic method. Let $(B_t)_{t \geq 0}$ be the three dimensional Brownian motion on a probability space $(\mathcal{W}, B(\mathcal{W}), P^x)$. Here P^x is the Wiener measure starting from x . The expectation with respect to P^x is simply denoted by $\mathbb{E}^x[\dots]$. Let $\mathcal{A}(F)$ be the Gaussian random process indexed by $F \in \oplus^3 L^2(\mathbb{R}^3)$ on a probability space $(Q, \mathcal{B}(Q), \mu)$ such that $\mathbb{E}_\mu[\mathcal{A}(F)] = 0$ and the covariance is given by $\mathbb{E}_\mu[\mathcal{A}(F)\mathcal{A}(G)] = \frac{1}{2} \sum_{\mu, \nu=1}^3 (\hat{F}_\mu, d_{\mu\nu} \hat{G}_\nu)$, where $d_{\mu\nu}(k) = \delta_{\mu\nu} - \frac{k_\mu k_\nu}{|k|^2}$. The unitary equivalence between $L^2(Q)$ and \mathcal{F} is established and under this equivalence it follows that for $F = F_1 \oplus F_2 \oplus F_3 \in \oplus^3 L^2(\mathbb{R}^3)$,

$$\mathcal{A}(F) \cong \frac{1}{\sqrt{2}} \sum_{\mu=1}^3 \sum_{j=1,2} \int e_\mu(k, j) (a^\dagger(k, j) \hat{F}_\mu(k) + a(k, j) \hat{F}_\mu(-k)) dk. \quad (3.1)$$

We set the right-hand side above as $A(F)$.

Proposition 3.1 *The Feynman-Kac type formula of e^{-tT_p} is given by*

$$(\Phi, e^{-tT_p} \Psi) = \int_{\mathbb{R}^3} dx \mathbb{E}^x[(\Phi(B_0), e^{-i\mathcal{A}(K)} \Psi(B_t))_{L^2(Q)}]. \quad (3.2)$$

Here

$$K = \oplus_{\mu=1}^3 \int_0^t (\tilde{\varphi}(\cdot - B_s) - \tilde{\varphi}(\cdot)) dB_s^\mu \quad (3.3)$$

with $\tilde{\varphi} = (\hat{\varphi}/\sqrt{\omega})^\sim$.

Proof: See [Hir00]. ■

Let

$$D_\infty = \cap_{\mu=1}^3 C^\infty(p_\mu) \cap C^\infty(N). \quad (3.4)$$

The range of T_p restricted on D_∞ is denoted by

$$\mathcal{D} = T_p D_\infty.$$

Lemma 3.2 *It follows that $\mathcal{D} \subset D_\infty \subset C^\infty(T_p)$. In particular $\mathcal{D} \subset D(a(F)) \cap D(\sqrt{T_p})$, and $a(F)\mathcal{D} \subset D(\sqrt{T_p})$.*

Proof: Since $a^\sharp(F)$ leaves $C^\infty(N)$ invariant, and $A_\mu(x)\Phi$ for $\Phi \in \mathcal{D}$ is infinitely differentiable with respect to x by virtue of the fact that $\hat{\varphi}$ has a compact support, it follows that $\mathcal{D} \subset D_\infty$. Furthermore we can check that $T_p : D_\infty \rightarrow D_\infty$, then $D_\infty \subset C^\infty(T_p)$ follows. Note that $D(a(F)) \supset D(N^{\frac{1}{2}})$ and $D(\sqrt{T_p}) \supset D(T_p)$. Then $\mathcal{D} \subset D(a(F)) \cap D(\sqrt{T_p})$ holds true, and since $a(F)\mathcal{D} \subset D_\infty$, $a(F)\mathcal{D} \subset D(T_p)$ and hence $a(F)\mathcal{D} \subset D(\sqrt{T_p})$ follows. ■

By Lemma 3.2, on \mathcal{D} operator $\sqrt{T_p}a(F)$ is well defined, but it is not clear whether $a(F)\sqrt{T_p}$ can be defined on \mathcal{D} or not. Hence we shall prove that

- (1) \mathcal{D} is dense,
(2) $\sqrt{T_p}\mathcal{D} \subset D(N)$.

Statement (2) guarantees that $a(F)\sqrt{T_p}$ is well defined on \mathcal{D} because of the fact that $D(a(F)) \supset D(N^{\frac{1}{2}}) \supset D(N)$. Hence together with Lemma 3.2 we can conclude that commutator $[a(F), \sqrt{T_p}]$ is well defined on \mathcal{D} . In order to prove (1) and (2), we prepare several lemmas. We have

$$Ne^{-i\mathcal{A}(K)}\Phi = e^{-i\mathcal{A}(K)}(N - \Pi(K) - \xi_K)\Phi,$$

where $\Pi(K)$ denotes the conjugate momentum of $\mathcal{A}(K)$, $\Pi(K) = i[N, \mathcal{A}(K)]$, and

$$\xi_K = \frac{1}{2} \sum_{\mu, \nu=1}^3 (\hat{K}_\mu, \delta_{\mu\nu}^\perp \hat{K}_\nu)_{L^2(\mathbb{R}^3)}$$

is a stochastic process. Note that under the identification $L^2(Q) \cong \mathcal{F}$,

$$\Pi(K) \cong \frac{i}{\sqrt{2}} \sum_{\mu=1}^3 \sum_{j=1,2} \int e_\mu(k, j) (a^\dagger(k, j) \hat{K}_\mu(k) - a(k, j) \hat{K}_\mu(-k)) dk \quad (3.5)$$

and $\hat{K}_\mu = \int_0^t \phi_\omega(k) (e^{-ikB_s} - 1) dB_s^\mu$ is $L^2(\mathbb{R}^3)$ -valued stochastic integral. We set the right-hand side above as $\pi(K)$. Let $P_\mu = p_\mu \otimes \mathbb{1} + \mathbb{1} \otimes P_{f\mu}$, $\mu = 1, 2, 3$, be the total momentum. We can also see the commutation relation between $P_{f\nu}$ and $e^{-i\mathcal{A}(K)}$, which is given by

$$P_{f\nu} e^{-i\mathcal{A}(K)} \Phi = e^{-i\mathcal{A}(K)} (P_{f\nu} - \Pi_\nu(K) - \xi_K^\nu) \Phi.$$

Here $\Pi_\nu(K)$ is defined by $\Pi_\nu(K) = i[P_{f\nu}, \mathcal{A}(K)]$, and

$$\xi_K^\nu = \frac{1}{2} \sum_{\mu, \rho=1}^3 (k_\nu \hat{K}_\mu, \delta_{\mu\rho}^\perp \hat{K}_\rho)$$

is also a stochastic process. Note that

$$\Pi_\nu(F) \cong \frac{i}{\sqrt{2}} \sum_{\mu=1}^3 \sum_{j=1,2} \int k_\nu e_\mu(k, j) \left(a^\dagger(k, j) \hat{F}_\mu(k) - a(k, j) \hat{F}_\mu(-k) \right) dk. \quad (3.6)$$

We set the right-hand side above as $\pi_\nu(K)$.

Lemma 3.3 *Let K be $\oplus^3 L^2(\mathbb{R}^3)$ -valued stochastic integral given by (3.3). Then (1) and (2) below follow:*

- (1) *Let $k \in \mathbb{N}$. Then there exists a polynomial $P_k = P_k(x)$ of degree k such that*

$$\|(N - \pi(K) - \xi_K)^k \Phi\|_{\mathcal{F}} \leq P_k(\xi_K) \|(N + \mathbb{1})^k \Phi\|_{\mathcal{F}}.$$

(2) Let $1 \leq \mu_1, \dots, \mu_n \leq 3$. Then there exists a polynomial $Q_n = Q_n(x_1, \dots, x_n)$ of degree n such that

$$\|(N - \pi_{\mu_1}(K) - \xi_K^{\mu_1}) \cdots (N - \pi_{\mu_n}(K) - \xi_K^{\mu_n})\Phi\|_{\mathcal{F}} \leq Q_n(\xi_K^{\mu_1}, \dots, \xi_K^{\mu_n})\|(N + \mathbb{1})^n \Phi\|_{\mathcal{F}}.$$

Proof: Let us show (1) by an induction. For $k = 1$ it can be seen that $\|(N - \pi(K) - \xi_K)\Phi\| \leq \|N\Phi\| + \|\pi(K)\Phi\| + |\xi_K|\|\Phi\|$. Since $\|\pi(K)\Phi\| \leq C\sqrt{\xi_K}\|(N + \mathbb{1})^{\frac{1}{2}}\Phi\|$, (1) follows. Suppose that (1) is true for $k = 1, \dots, n$. Then we have

$$\begin{aligned} & \|(N - \pi(K) - \xi_K)^{n+1}\Phi\| \\ & \leq \|(N - \pi(K) - \xi_K)^n N\Phi\| + \|(N - \pi(K) - \xi_K)^n \pi(K)\Phi\| + \|(N - \pi(K) - \xi_K)^n \xi_K \Phi\|. \end{aligned}$$

By the assumption of the induction it is trivial to see that

$$\begin{aligned} \|(N - \pi(K) - \xi_K)^n N\Phi\| & \leq P_n(\xi_K)\|(N + \mathbb{1})^{n+1}\Phi\|, \\ \|(N - \pi(K) - \xi_K)^n \xi_K \Phi\| & \leq P_n(\xi_K)\xi_K\|(N + \mathbb{1})^{n+1}\Phi\|. \end{aligned}$$

We can also see that

$$\begin{aligned} \|(N - \pi(K) - \xi_K)^n \pi(K)\Phi\| & \leq P_k(\xi_K)\|(N + \mathbb{1})^n \pi(K)\Phi\| \\ & \leq P_k(\xi_K)(\|(N + \mathbb{1})^{n-1} \pi(K)(N + \mathbb{1})\Phi\| + \|(N + \mathbb{1})^{n-1} A(K)\Phi\|) \end{aligned}$$

and

$$\begin{aligned} \|(N + \mathbb{1})^m \pi(K)\Phi\| & \leq c\sqrt{\xi_K}\|(N + \mathbb{1})^{m+1}\Phi\|, \\ \|(N + \mathbb{1})^m A(K)\Phi\| & \leq c\sqrt{\xi_K}\|(N + \mathbb{1})^{m+1}\Phi\| \end{aligned}$$

with some constant c . Then

$$\|(N - \pi(K) - \xi_K)^n \pi(K)\Phi\| \leq CP(\xi_K)\sqrt{\xi_K}\|(N + \mathbb{1})^{n+1}\Phi\|.$$

Then statement (1) follows. Statement (2) can be similarly proven. ■

Lemma 3.4 Suppose Assumption 2.1. Then $e^{-tT_p}D_\infty \subset D_\infty$. In particular $T_p|_{D_\infty}$ is essentially self-adjoint and \mathcal{D} is dense.

Proof: Let $\Psi \in D_\infty$ be arbitrary. It is enough to show that $e^{-tT_p}\Psi \in D(N^n) \cap D(p_{\mu_1} \cdots p_{\mu_m})$ for arbitrary n and $1 \leq \mu_1, \dots, \mu_m \leq 3$. In order to do that we show bounds below for arbitrary $\Phi_1 \in D(N^n)$ and $\Phi_2 \in D(p_{\mu_1} \cdots p_{\mu_m})$:

$$|(N^n \Phi_1, e^{-tT_p}\Psi)| \leq C\|\Phi_1\|, \tag{3.7}$$

$$|(p_{\mu_1} \cdots p_{\mu_m} \Phi_2, e^{-tT_p}\Psi)| \leq C\|\Phi_2\|. \tag{3.8}$$

By the Feynman-Kac formula and the equivalence $\Pi(K) \cong \pi(K)$, we have

$$\begin{aligned} |(N^n \Phi_1, e^{-tT_p} \Psi)| &= \left| \int_{\mathbb{R}^3} dx \mathbb{E}^x[(N^n \Phi_1(B_0), e^{-i\mathcal{A}(K)} \Psi(B_t))] \right| \\ &= \left| \int_{\mathbb{R}^3} dx \mathbb{E}^x[(\Phi_1(B_0), e^{-i\mathcal{A}(K)} (N - \Pi(K) - \xi_K)^n \Psi(B_t))] \right|. \end{aligned}$$

Using bounds shown in Lemma 3.3 we have

$$|(N^n \Phi_1, e^{-tT_p} \Psi)| \leq \int_{\mathbb{R}^3} dx \|\Phi_1(x)\| (\mathbb{E}^x[|P_n(\xi_K)|^2])^{\frac{1}{2}} (\mathbb{E}^x[\|(N + \mathbb{1})^n \Psi(B_t)\|])^{\frac{1}{2}}.$$

By the BDG inequality

$$\mathbb{E}^x[|\xi_K|^m] \leq ct^m \quad (3.9)$$

with some constant c proved in [Hir00, Theorem 4.5], we can then get $(\mathbb{E}^x[|P_n(\xi_K)|^2])^{\frac{1}{2}} < Ct^n$ for some constant C , hence

$$|(N^n \Phi_1, e^{-tT_p} \Psi)| \leq C \|\Phi_1\| \|(N + \mathbb{1})^n \Psi\|.$$

Next we estimate (3.8). Note that $[P_\mu, e^{-i\mathcal{A}(K)}] = 0$ for $\mu = 1, 2, 3$. We have

$$\begin{aligned} p_\mu e^{-i\mathcal{A}(K)} &= (P_\mu - P_{f\mu}) e^{-i\mathcal{A}(K)} = -P_{f\mu} e^{-i\mathcal{A}(K)} + e^{-i\mathcal{A}(K)} P_\mu \\ &= e^{-i\mathcal{A}(K)} (p_\mu + \Pi_\mu(K) + \xi_K^\mu). \end{aligned}$$

Hence

$$p_{\mu_1} \cdots p_{\mu_m} e^{-i\mathcal{A}(K)} = e^{-i\mathcal{A}(K)} (p_{\mu_1} + \Pi_{\mu_1}(K) + \xi_K^{\mu_1}) \cdots (p_{\mu_m} + \Pi_{\mu_m}(K) + \xi_K^{\mu_m}).$$

Then we have

$$\begin{aligned} |(p_{\mu_1} \cdots p_{\mu_m} \Phi_2, e^{-tT_p} \Psi)| &= \left| \int_{\mathbb{R}^3} dx \mathbb{E}^x[(p_{\mu_1} \cdots p_{\mu_m} \Phi_2(B_0), e^{-i\mathcal{A}(K)} \Psi(B_t))] \right| \\ &= \left| \int_{\mathbb{R}^3} dx \mathbb{E}^x[(\Phi_2(B_0), e^{-i\mathcal{A}(K)} (p_{\mu_1} + \Pi_{\mu_1}(K) + \xi_K^{\mu_1}) \cdots (p_{\mu_m} + \Pi_{\mu_m}(K) + \xi_K^{\mu_m}) \Psi(B_t))] \right|. \end{aligned}$$

Using again bounds shown in Lemma 3.3 we have

$$|(p_{\mu_1} \cdots p_{\mu_m} \Phi_2, e^{-tT_p} \Psi)| \leq \int_{\mathbb{R}^3} dx \|\Phi(x)\| (\mathbb{E}^x[|Q_m(\xi_K^{\mu_1}, \dots, \xi_K^{\mu_m})|^2])^{\frac{1}{2}} (\mathbb{E}^x[\|(N + \mathbb{1})^m \Phi\|^2])^{\frac{1}{2}}.$$

Thus in a similar manner to (3.7), the BDG-inequality (3.9) yields that

$$(\mathbb{E}[|Q_m(\xi_K^{\mu_1}, \dots, \xi_K^{\mu_m})|^2])^{\frac{1}{2}} \leq Ct^m,$$

and we can show (3.8). Then D_∞ is an invariant domain of e^{-tT_p} , which implies that $T_p|_{D_\infty}$ is essentially self-adjoint and thus $\mathcal{D} = T_p D_\infty$ is dense. \blacksquare

Let us set

$$R_{t^2} = (T_p + t^2)^{-1}.$$

Lemma 3.5 *Suppose Assumption 2.1. Let $\Psi \in \mathcal{D}$. Then $\sqrt{T_p}\Psi \in D(N)$.*

Proof: We shall show that

$$|(N\Phi, \sqrt{T_p}\Psi)| \leq C\|\Phi\| \quad (3.10)$$

for any $\Phi \in D(N)$ with some constant C independent of Φ . In order to show (3.10) we again apply Feynman-Kac formula for e^{-tT_p} . By the definition of $\sqrt{T_p}$ we have

$$(N\Phi, \sqrt{T_p}\Psi) = \frac{2}{\pi} \int_0^\infty (N\Phi, R_{\lambda^2} T_p \Psi) d\lambda. \quad (3.11)$$

We divide integral (3.11) as $\int_0^\infty \cdots d\lambda = \int_0^1 \cdots d\lambda + \int_1^\infty \cdots d\lambda$. We estimate $\int_1^\infty \cdots d\lambda$. Fix λ . We have

$$(N\Phi, R_{\lambda^2} T_p \Psi) = \int_0^\infty dt \int_{\mathbb{R}^3} dx \mathbb{E}^x[(N\Phi(B_0), e^{-i\mathcal{A}(K)} F(B_t))] e^{-t\lambda^2}, \quad (3.12)$$

where we set $F = T_p\Psi$, and use the identity $R_{\lambda^2} = \int_0^\infty e^{-tT_p - t\lambda^2} dt$. Since $e^{-i\mathcal{A}(K)} F(B_t) \in D(N)$, we have

$$\begin{aligned} & \int_1^\infty d\lambda \int_0^\infty dt \int_{\mathbb{R}^3} dx \mathbb{E}^x[(N\Phi(B_0), e^{-i\mathcal{A}(K)} F(B_t))] e^{-t\lambda^2} \\ &= \int_1^\infty d\lambda \int_0^\infty dt \int_{\mathbb{R}^3} dx \mathbb{E}^x[(\Phi(B_0), e^{-i\mathcal{A}(K)} (N - \Pi(K) - \xi_K) F(B_t))] e^{-t\lambda^2}. \end{aligned}$$

We estimate integrands as

$$\begin{aligned} |(\Phi(B_0), e^{-i\mathcal{A}(K)} NF(B_t))| &\leq \|\Phi(x)\| \|NF(B_t)\|, \\ |(\Phi(B_0), e^{-i\mathcal{A}(K)} \Pi(K) F(B_t))| &\leq \|\Phi(x)\| \|(N + \mathbb{1})^{\frac{1}{2}} F(B_t)\| \sqrt{\xi_K}, \\ |(\Phi(B_0), e^{-i\mathcal{A}(K)} \xi_K F(B_t))| &\leq \|\Phi(x)\| \|F(B_t)\| |\xi_K|. \end{aligned}$$

By BDG inequality (3.9) we can derive that

$$\begin{aligned} \mathbb{E}^x[\|\Phi(x)\| \|(N + \mathbb{1})^{\frac{1}{2}} F(B_t)\| \sqrt{\xi_K}] &\leq \|\Phi(x)\| (\mathbb{E}^x[\|(N + \mathbb{1})^{\frac{1}{2}} F(B_t)\|^2])^{\frac{1}{2}} (\mathbb{E}^x[\xi_K])^{\frac{1}{2}} \\ &\leq \sqrt{ct} \|\Phi(x)\| (\mathbb{E}^x[\|(N + \mathbb{1})^{\frac{1}{2}} F(B_t)\|^2])^{\frac{1}{2}} \end{aligned}$$

and

$$\mathbb{E}^x[(\Phi(B_0), e^{-i\mathcal{A}(K)} \xi_K F(B_t))] \leq \|\Phi(x)\| \mathbb{E}^x[\|F(B_t)\|] tc.$$

Together with them we have

$$\begin{aligned} & \int_1^\infty d\lambda \int_0^\infty dt \int_{\mathbb{R}^3} dx \mathbb{E}^x[(N\Phi(B_0), e^{-i\mathcal{A}(K)} F(B_t))] e^{-t\lambda^2} \\ &< c \int_1^\infty d\lambda \int_0^\infty dt e^{-t\lambda^2} \|\Phi\| \left(t\|F\| + \sqrt{t} \|(N + \mathbb{1})^{\frac{1}{2}} F\| + \|NF\| \right). \end{aligned}$$

Note that $\int_0^\infty e^{-t\lambda^2} t dt = \lambda^{-4}$ and $\int_1^\infty \lambda^{-4} d\lambda < \infty$. Thus it follows that

$$\int_1^\infty (N\Phi, R_{\lambda^2} T_p \Psi) d\lambda \leq C \|\Phi\| (\|T_p \Psi\| + \|N^{\frac{1}{2}} T_p \Psi\| + \|N T_p \Psi\|) \quad (3.13)$$

with some constant C . Next we estimate $\int_0^1 \dots d\lambda$.

$$\int_0^1 (N\Phi, R_{\lambda^2} T_p \Psi) d\lambda = \int_0^1 (N\Phi, \Psi) d\lambda + \int_0^1 (N\Phi, -\lambda^2 R_{\lambda^2} \Psi) d\lambda.$$

Since $\Psi \in \mathcal{D}$, there exists $\phi \in D_\infty$ such that $\Psi = T_p \phi$. Then

$$\int_0^1 (N\Phi, -\lambda^2 R_{\lambda^2} \Psi) d\lambda = - \int_0^1 \lambda^2 (N\Phi, \phi) d\lambda + \int_0^1 \lambda^4 (N\Phi, R_{\lambda^2} \phi) d\lambda. \quad (3.14)$$

It is trivial to see that

$$\left| \int_0^1 \lambda^2 (N\Phi, \phi) d\lambda \right| \leq \frac{1}{3} \|\Phi\| \|N\phi\|. \quad (3.15)$$

In a similar manner to (3.13) we can see that

$$|(N\Phi, R_{\lambda^2} \phi)| \leq \int_0^\infty dt e^{-t\lambda^2} \|\Phi\| (t\|\phi\| + \sqrt{t}\|(N + \mathbb{1})^{\frac{1}{2}} \phi\| + \|N\phi\|).$$

We note that $\int_0^\infty dt \lambda^4 t e^{-t\lambda^2} = \int_0^\infty u e^{-u} du = c_1$, $\int_0^\infty dt \lambda^4 \sqrt{t} e^{-t\lambda^2} = \lambda \int_0^\infty \sqrt{u} e^{-u} du = \lambda c_2$ and $\int_0^\infty dt \lambda^4 e^{-t\lambda^2} = \lambda^2 \int_0^\infty e^{-u} du = \lambda^2 c_2$. Hence it follows that

$$\int_0^1 \lambda^4 (N\Phi, R_{\lambda^2} \Psi) d\lambda \leq C \|\Phi\| (\|N T_p \phi\| + \|N\phi\| + \|\phi\| + \|(N + \mathbb{1})^{\frac{1}{2}} \phi\|) \quad (3.16)$$

with some constant C . Then from (3.13), (3.15) and (3.16), (3.10) follows. \blacksquare

Lemma 3.6 *Suppose Assumption 2.1. Then $\mathcal{D} \subset D(a(F)\sqrt{T_p}) \cap D(\sqrt{T_p}a(F))$.*

Proof: $\mathcal{D} \subset D(\sqrt{T_p}a(F))$ follows from Lemma 3.2 and $\mathcal{D} \subset D(a(F)\sqrt{T_p})$ from Lemma 3.5. \blacksquare

3.2 Commutator estimates and number operator bounds

Let $m > 0$ throughout this section. In this section we estimate $\|N^{\frac{1}{2}} \Phi_m\|$ uniformly in $m > 0$. In order to do this we apply or suitably modify the method developed in [Hir05]. Let $D \subset D(A) \cap D(B)$. The weak commutator $[A, B]_W^D(\Phi, \Psi)$ is the sesquilinear form defined by

$$[A, B]_W^D(\Phi, \Psi) = (A\Phi, B\Psi) - (B\Phi, A\Psi)$$

for $\Phi, \Psi \in D$.

Proposition 3.7 *Suppose (1)-(3) below:*

- (1) *There exists an operator $B_j(k) : \mathcal{F} \rightarrow \mathcal{F}$ for each $k \in \mathbb{R}^3$, $j = 1, 2$, such that $D(B_j(k)) \supset D(H_m^R)$ for almost everywhere k , and*

$$[a(f, j), \sqrt{T_p}]_W^{D(H_m^R)}(\Psi, \Phi) = \int_{\mathbb{R}^3} f(k)(\Psi, B_j(k)\Phi)dk.$$

- (2) *Let $K = \cup_{j=1}^3 \{k = (k_1, k_2, k_3) | k_j = 0\}$. For $f \in C_0^\infty(\mathbb{R}^3 \setminus K)$ and $\Psi \in D(H_m)$ it follows that*

$$\int_{\mathbb{R}^3} dk f(k)(\Psi, e^{-it(H_m^R - E_m + \omega(k))} B_j(k)\Phi_m) \in L^1([0, \infty), dt).$$

- (3) $\|B_j(\cdot)\Phi_m\| \in L^2(\mathbb{R}^3)$.

Then $\Phi_m \in D(N^{\frac{1}{2}})$ if and only if $\int_{\mathbb{R}^3} \|(H_m^R - E_m + \omega(k))^{-1} B_j(k)\Phi_m\|^2 dk < \infty$. Furthermore when $\Phi_m \in D(N^{\frac{1}{2}})$, it follows that

$$\|N^{\frac{1}{2}}\Phi_m\|^2 = \int_{\mathbb{R}^3} \|(H_m^R - E_m + \omega(k))^{-1} B_j(k)\Phi_m\|^2 dk.$$

Proof: See [Hir05, Example 2.4 and Theorem 2.9] and Appendix A. The statement (1) is given as (B2) in [Hir05], (2) as (B3) and (3) as (B4). ■

Suppose (1), (2) and (3) in Proposition 3.7. Then we define $T_{gj} : L^2(\mathbb{R}^3) \rightarrow \mathcal{H}$ by

$$T_{gj}f = \int_{\mathbb{R}^3} f(k)(H_m^R - E_m + \omega(k))^{-1} B_j(k)\Phi_m dk, \quad j = 1, 2, 3,$$

with the domain $D(T_{gj}) = \{f \in L^2(\mathbb{R}^3) | \|\int_{\mathbb{R}^3} f(k)(H_m^R - E_m + \omega(k))^{-1} B_j(k)\Phi_m\| < \infty\}$.

Proposition 3.8 *Suppose (1), (2) and (3) in Proposition 3.7. Then (1)*

$$\int_{\mathbb{R}^3} \|(H_m^R - E_m + \omega(k))^{-1} B_j(k)\Phi_m\| dk < \infty.$$

(2) $a(f, j)\Phi_m = -T_{gj}f$ for $f, f/\sqrt{\omega} \in L^2(\mathbb{R}^3)$. (3) $\Phi_m \in D(N^{\frac{1}{2}})$ if and only if T_{gj} is a Hilbert-Schmidt operator. (4) If T_{gj} is a Hilbert-Schmidt operator. Then the Hilbert-Schmidt norm of T_{gj} is given by

$$\text{Tr}(T_{gj}^* T_{gj}) = \int_{\mathbb{R}^3} \|(H_m^R - E_m + \omega(k))^{-1} B_j(k)\varphi_m\|^2 dk.$$

Proof: See [Hir05, Lemmas 2.7 and 2.8]. ■

We note that $\omega \in C^\infty(\mathbb{R}^3 \setminus \{K\})$. We set

$$T_j(k) = e(k, j) \cdot (p - A_R). \tag{3.17}$$

For each $k \in \mathbb{R}^3$ let us define the operator $I_j(k)$ by $I_j(k) = \int_0^\infty I_j(k, t) dt$, where

$$I_j(k, t) = t^2 R_{t^2} T_j(k) (e^{-ikx} - 1) R_{t^2} \frac{1}{\langle x \rangle^2}.$$

Let

$$C_j(k) = \frac{4}{\pi} \phi_\omega(k) I_j(k) + \rho_j(k) \frac{1}{\langle x \rangle^2} \quad (3.18)$$

and

$$\rho_j(k) = -i \sqrt{\omega(k)} \hat{\varphi}(k) e(k, j) \cdot x.$$

Lemma 3.9 *Suppose Assumption 2.1. Let $f \in L^2(\mathbb{R}^3)$. Then $C_j(k)$ is a bounded operator for each $k \in \mathbb{R}^3$ with*

$$\|C_j(k)\| \leq C(|k| + |k|^2) \phi_\omega(k) \quad (3.19)$$

and

$$[H_{\text{int}}, a(f, j)]_W^{\text{D}(H_m^{\text{R}})}(\Phi, \Psi) = \int_{\mathbb{R}^3} dk f(k) (\Phi, C_j(k) \langle x \rangle^2 \Psi) \quad (3.20)$$

for $\Phi, \Psi \in \text{D}(H_m^{\text{R}})$ with $\Psi \in \text{D}(\langle x \rangle^2)$, and $\|C_j(\cdot)\Psi\| \in L^2(\mathbb{R}^3)$.

Proof: Let us consider $[H_{\text{int}}, a(f, j)] = [\sqrt{T_p}, a(f, j)] + [h, a(f, j)]$ on \mathcal{D} . We have

$$[h, a(f, j)] = \int_{\mathbb{R}^3} \rho_j(k) f(k) dk.$$

On \mathcal{D} , we also have

$$[H_{\text{int}}, a(f, j)]_W^{\text{D}(H_m^{\text{R}})}(\Phi, \Psi) = (\Phi, [\sqrt{T_p}, a(f, j)] \Psi) + \int_{\mathbb{R}^3} f(k) (\Phi, \rho_j(k) \Psi) dk$$

and

$$(\Phi, [\sqrt{T_p}, a(f, j)] \Psi) = \frac{2}{\pi} \int_0^\infty (\Phi, [T_p R_{t^2}, a(f, j)] \Psi) dt = -\frac{2}{\pi} \int_0^\infty t^2 (\Phi, R_{t^2} [a(f, j), T_p] R_{t^2} \Psi) dt.$$

The commutator $[a(f, j), T_p]$ is computed on $R_{t^2} \mathcal{D}$ as

$$(R_{t^2} \Phi, [a(f, j), T_p] R_{t^2} \Psi) = \sqrt{2} \int_{\mathbb{R}^3} f(k) \phi_\omega(k) (R_{t^2} \Phi, (e^{-ikx} - 1) T_j(k) R_{t^2} \Psi) dk.$$

Since the Coulomb gauge condition $k \cdot e(k, j) = 0$, we have $T_j(k) e^{-ikx} = e^{-ikx} T_j(k)$. Then

$$(\Phi, [\sqrt{T_p}, a(f, j)] \Psi) = \frac{4}{\pi} \int_0^\infty dt \int_{\mathbb{R}^3} dk f(k) (\Phi, I_j(k, t) \phi_\omega(k) \langle x \rangle^2 \Psi).$$

It is also shown in Lemma 3.10 that

$$\int_0^\infty |(\Phi, I_j(k, t) \langle x \rangle^2 \Psi)| dt \leq C(|k| + |k|^2) \|\Phi\| \langle x \rangle^2 \Psi \| \quad (3.21)$$

with a constant C independent of m and k , and $f(k)(|k| + |k|^2)\phi_\omega(k)$ is integrable by the fact that ϕ_ω has a compact support. By Fubini's lemma, we can see that

$$(\Phi, [\sqrt{T_p}, a(f, j)]\Psi) = \int_{\mathbb{R}^3} f(k) dk \left(\Phi, \frac{4}{\pi} I_j(k) \phi_\omega(k) \langle x \rangle^2 \Psi \right).$$

Hence (3.20) follows. We can see in Lemma 3.10 that $I_j(k)$ is bounded with $\|I_j(k)\| \leq C(|k| + |k|^2)$. On the other hand, $\|\rho_j(k) \frac{1}{\langle x \rangle^2}\| \leq \omega(k) |\phi_\omega(k)|$. Then for almost every $k \in \mathbb{R}^3$, $C_j(k)$ is bounded and (3.19) follows. In particular $\|C_j(\cdot)\Psi\| \in L^2(\mathbb{R}^3)$. Then the proof is complete. \blacksquare

Lemma 3.10 *For each $k \in \mathbb{R}^3$, $I_j(k)$ is a bounded operator such that*

$$|(\Phi, I_j(k) \Psi)| \leq C(|k| + |k|^2) \|\Phi\| \|\Psi\|.$$

Proof: For all $\Psi \in \mathcal{H}$, it holds that $\|T_j(k) R_{t^2} \Psi\| \leq C \|\sqrt{T_p} R_{t^2} \Psi\|$. Set

$$\begin{aligned} I_{1,j}(k, \Psi, \Phi) &= \int_0^1 |(\Psi, I_j(k, t) \Phi)| dt, \\ I_{2,j}(k, \Psi, \Phi) &= \int_1^\infty |(\Psi, I_j(k, t) \Phi)| dt \end{aligned}$$

for $\Psi, \Phi \in \mathcal{D}$. By Schwarz's inequality,

$$\begin{aligned} I_{1,j}(k, \Psi, \Phi) &\leq \int_0^1 dt t^2 \|T_j(k) R_{t^2} \Psi\| \left\| (e^{-ikx} - 1) R_{t^2} \frac{1}{\langle x \rangle^2} \Phi \right\| \\ &\leq C|k| \int_0^1 dt t^2 \left\| \sqrt{T_p} R_{t^2} \Psi \right\| \left\| |x| R_{t^2} \frac{1}{\langle x \rangle^2} \Phi \right\| \\ &\leq C|k| \left(\int_0^1 dt t \left\| \sqrt{T_p} R_{t^2} \Psi \right\|^2 \right)^{\frac{1}{2}} \left(\int_0^1 t^3 dt \left\| |x| R_{t^2} \frac{1}{\langle x \rangle^2} \Phi \right\|^2 \right)^{\frac{1}{2}} \\ &\leq C|k| \|\Psi\| \left(\int_0^1 t^3 dt \left\| |x| R_{t^2} \frac{1}{\langle x \rangle^2} \Phi \right\|^2 \right)^{\frac{1}{2}}. \end{aligned} \quad (3.22)$$

Here we used the estimate:

$$\int_0^1 dt t \left\| \sqrt{T_p} R_{t^2} \Psi \right\|^2 = \int_0^\infty dE_\lambda \int_0^1 dt \frac{\lambda t}{(\lambda + t^2)^2} = \frac{1}{2} \int_0^\infty \frac{1}{\lambda + 1} dE_\lambda \leq \frac{1}{2}, \quad (3.23)$$

where dE_λ denotes the spectral measure of T_p with respect to Ψ . The diamagnetic inequality yields that

$$\left\| |x| R_{t^2} \frac{1}{\langle x \rangle^2} \Phi \right\| \leq \left\| |x| \frac{1}{t^2 + |p|^2} \frac{1}{\langle x \rangle^2} |\Phi| \right\|. \quad (3.24)$$

Then we have by (3.22)

$$I_{1,j}(k, \Psi, \Phi) \leq C|k| \|\Psi\| \sqrt{(|\Phi|, Z|\Phi|)},$$

where $Z : L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3)$ is the operator defined by

$$Zf = \frac{1}{\langle x \rangle^2} \int_0^1 dt t^3 \frac{1}{t^2 + |p|^2} |x|^2 \frac{1}{t^2 + |p|^2} \frac{1}{\langle x \rangle^2} f.$$

We shall show that Z is bounded. Let $\mathcal{W} = \{u \in L^2(\mathbb{R}^3) | \hat{u} \in C_0^\infty(\mathbb{R}^3 \setminus 0)\}$. \mathcal{W} is a dense subspace of $L^2(\mathbb{R}^3)$. We have

$$\frac{1}{t^2 + |p|^2} |x|^2 \frac{1}{t^2 + |p|^2} = x \frac{1}{(t^2 + |p|^2)^2} x - 2ixp \frac{1}{(t^2 + |p|^2)^3} + 2i \frac{1}{(t^2 + |p|^2)^3} px + \frac{4|p|^2}{(t^2 + |p|^2)^4} \quad (3.25)$$

on \mathcal{W} . For $u \in \mathcal{W}$, we set $v = \frac{1}{\langle x \rangle^2} u$. Then

$$\begin{aligned} |(u, Zu)| &\leq \int_0^1 (v, x \frac{1}{(t^2 + |p|^2)^2} xv) t^3 dt \\ &\quad + 4 \left| \Re \int_0^1 (v, i \frac{1}{(t^2 + |p|^2)^3} p \cdot xv) t^3 dt \right| + 4 \int_0^1 (v, \frac{1}{(t^2 + |p|^2)^4} |p|^2 v) t^3 dt. \end{aligned}$$

Note that

$$\begin{aligned} \int_0^1 \frac{t^3 dt}{(t^2 + |p|^2)^2} &\leq \frac{1}{|p|^2}, \quad \int_0^1 \frac{t^3 dt}{(t^2 + |p|^2)^3} = \frac{1}{4|p|^2(1 + |p|^2)}, \\ \int_0^1 \frac{t^3 dt}{(t^2 + |p|^2)^4} &= \frac{1}{12|p|^4(1 + |p|^2)^2} + \frac{1}{4|p|^2(1 + |p|^2)^3}. \end{aligned}$$

Thus we have

$$\begin{aligned} |(u, Zu)| &\leq (xv, \frac{1}{|p|^2} xv) + \left| \left(v, \frac{1}{|p|^2(1 + |p|^2)^2} p \cdot xv \right) \right| + \left| \left(v, \frac{1}{3|p|^2(1 + |p|^2)^2} v \right) \right| + \left| \left(v, \frac{1}{(1 + |p|^2)^2} v \right) \right| \\ &\leq \| |p|^{-1} x \frac{1}{\langle x \rangle^2} u \| + \| |p|^{-\frac{1}{2}} \frac{1}{\langle x \rangle^2} u \| \| |p|^{-\frac{1}{2}} |x| \frac{1}{\langle x \rangle^2} u \| + \frac{1}{3} \| |p|^{-1} \frac{1}{\langle x \rangle^2} u \| + \| \frac{1}{\langle x \rangle^2} u \|^2. \end{aligned}$$

By the Hardy-Rellich inequality[Yaf99], we have for all $u \in \mathcal{W}$

$$|(u, Zu)| \leq C \left(\left\| \frac{|x|^2}{\langle x \rangle^2} u \right\| + \left\| \frac{|x|^{\frac{1}{2}}}{\langle x \rangle^2} u \right\| \left\| \frac{|x|^{\frac{3}{2}}}{\langle x \rangle^2} u \right\| + \frac{1}{3} \left\| \frac{|x|}{\langle x \rangle^2} u \right\| \right) + \|u\|^2 \leq C \|u\|^2. \quad (3.26)$$

Thus Z is a bounded operator on $L^2(\mathbb{R}^3)$. Then we obtain that

$$\|I_{1,j}(k, \Psi, \Phi)\| \leq C|k| \|\Psi\| \|\Phi\|. \quad (3.27)$$

Next we estimate $I_{2,j}(k, \Psi, \Phi)$. Set $T_{p+k} = e^{-ikx} T_p e^{ikx} = (p+k - A_R)^2$. Note that

$$\begin{aligned} (e^{-ikx} - 1)R_{t^2} &= R_{t^2}^{(k)}(e^{-ikx} - 1) + R_{t^2}(T_p - T_{p+k})R_{t^2}^{(k)}, \\ T_p - T_{p+k} &= -2Y(k) - |k|^2, \end{aligned}$$

where $R_{t^2}^{(k)} = (t^2 + T_{p+k})^{-1}$ and $Y(k) = k \cdot (p - A_R)$. Then

$$(e^{-ikx} - 1)R_{t^2} = R_{t^2}^{(k)}(e^{-ikx} - 1) - 2R_{t^2}Y(k)R_{t^2}^{(k)} - |k|^2 R_{t^2}R_{t^2}^{(k)} \quad (3.28)$$

which decomposition is often used in what follows. Thus

$$I_{2,j}(k, \Psi, \Phi) = I_2^{(1)}(k) + I_2^{(2)}(k) + I_2^{(3)}(k), \quad (3.29)$$

where

$$\begin{aligned} I_2^{(1)}(k) &= \int_1^\infty dt t^2 (\Psi, R_{t^2} T_j(k) R_{t^2}^{(k)} \frac{e^{-ikx} - 1}{\langle x \rangle^2} \Phi), \\ I_2^{(2)}(k) &= -2 \int_1^\infty dt t^2 (\Psi, R_{t^2} T_j(k) R_{t^2} Y(k) R_{t^2}^{(k)} \frac{1}{\langle x \rangle^2} \Phi), \\ I_2^{(3)}(k) &= -|k|^2 \int_1^\infty dt t^2 (\Psi, R_{t^2} T_j(k) R_{t^2} R_{t^2}^{(k)} \frac{1}{\langle x \rangle^2} \Phi). \end{aligned}$$

Let us estimate $I_2^{(1)}(k)$. Note that

$$\| |T_j(k)|^{\frac{1}{2}} \Psi \| \leq \| |(p+k - A_R)^2|^{\frac{1}{4}} \Psi \| + \sqrt{|k|} \|\Psi\| = \|T_{p+k}^{\frac{1}{4}} \Psi\| + \sqrt{|k|} \|\Psi\|.$$

Set $\phi = \frac{e^{-ikx}-1}{\langle x \rangle^2} \Phi$. By Schwarz's inequality we have

$$\begin{aligned} I_2^{(1)}(k) &\leq \left(\int_1^\infty dt t^2 \| |T_j(k)|^{\frac{1}{2}} R_{t^2} \Psi \|^2 \right)^{\frac{1}{2}} \left(\int_1^\infty dt t^2 \left\| |T_j(k)|^{\frac{1}{2}} R_{t^2}^{(k)} \phi \right\|^2 \right)^{\frac{1}{2}} \\ &\leq \left(\int_1^\infty dt t^2 \| T_p^{\frac{1}{4}} R_{t^2} \Psi \|^2 \right)^{\frac{1}{2}} \left(\left(\int_1^\infty dt t^2 \left\| T_{p+k}^{\frac{1}{4}} R_{t^2}^{(k)} \phi \right\|^2 \right)^{\frac{1}{2}} + \sqrt{|k|} \left(\int_1^\infty dt t^2 \left\| R_{t^2}^{(k)} \phi \right\|^2 \right)^{\frac{1}{2}} \right). \quad (3.30) \end{aligned}$$

Since for all $a > 0$, $\int_0^\infty \frac{t^2}{(t^2+a)^2} dt = \frac{\pi}{4\sqrt{a}}$, we see that

$$\begin{aligned} I_2^{(1)}(k) &\leq C \left(\int_0^\infty dE_\mu \int_1^\infty \frac{\mu^{\frac{1}{2}} t^2 dt}{(t^2 + \mu)^2} \right)^{\frac{1}{2}} \left(\left(\int_0^\infty d\tilde{E}_\mu \int_1^\infty \frac{\mu^{\frac{1}{2}} t^2 dt}{(t^2 + \mu)^2} \right)^{\frac{1}{2}} + \sqrt{|k|} \left(\int_0^\infty d\tilde{E}_\mu \int_1^\infty \frac{t^2 dt}{(t^2 + \mu)^2} \right)^{\frac{1}{2}} \right) \\ &\leq C(1 + \sqrt{|k|}) \|\Psi\| \|\Phi\|, \end{aligned} \quad (3.31)$$

where dE_μ and $d\tilde{E}_\mu$ are spectral measures of T_p and T_{p+k} with respect to Ψ and Φ , respectively. Thus we obtain that

$$I_2^{(1)}(k) \leq C|k|(1 + \sqrt{|k|}) \|\Psi\| \|\Phi\|. \quad (3.32)$$

Next let us estimate $I_2^{(2)}(k)$. Since $\|\sqrt{R_{t^2}}Y(k)\| \leq C|k|$, $\|T_j(k)\sqrt{R_{t^2}}\| \leq C$ and $\|R_{t^2}^{(k)} \frac{1}{\langle x \rangle^2} \Phi\| \leq \|\Phi\|/t^2$, by Schwarz's inequality,

$$I_2^{(2)}(k) \leq 2 \int_1^\infty dt t^2 \|R_{t^2} \Psi\| \|T_j(k)\sqrt{R_{t^2}}\| \|\sqrt{R_{t^2}}Y(k)\| \|R_{t^2}^{(k)} \frac{1}{\langle x \rangle^2} \Phi\| \leq C|k| \int_1^\infty \frac{dt}{t^2} \|\Psi\| \|\Phi\|.$$

Thus we obtain that

$$I_2^{(2)}(k) \leq C|k| \|\Psi\| \|\Phi\|. \quad (3.33)$$

Finally we estimate $I_2^{(3)}(k)$. By Schwarz's inequality again, it can be seen that

$$\begin{aligned} I_2^{(3)}(k) &\leq |k|^2 \int_1^\infty dt t^2 \|T_j(k)R_{t^2}\Psi\| \|R_{t^2}R_{t^2}^{(k)} \frac{1}{\langle x \rangle^2} \Phi\| \\ &\leq |k|^2 \int_1^\infty \frac{dt}{t^2} \|T_j(k)R_{t^2}\Psi\| \|\Phi\| \leq C|k|^2 \|\Psi\| \|\Phi\|. \end{aligned} \quad (3.34)$$

Then from (3.32), (3.33) and (3.34) it follows that

$$I_{2,j}(k, \Psi, \Phi) \leq C(|k| + |k|^2) \|\Psi\| \|\Phi\|. \quad (3.35)$$

By (3.27) and (3.35), the lemma is proven. ■

From the proof of Lemma 3.10 we can obtain a useful corollary used in Section 3.3.

Corollary 3.11 *There exists a constant C such that for any $\Phi \in \mathcal{H}$,*

$$\int_0^1 dt t^3 \left\| |x| R_{t^2} \frac{1}{\langle x \rangle^2} \Phi \right\|^2 \leq C \|\Phi\|^2, \quad (3.36)$$

$$\int_0^1 dt t^3 \left\| |x|^2 R_{t^2} \frac{1}{\langle x \rangle^2} \Phi \right\|^2 \leq C \| |x| \Phi \|^2. \quad (3.37)$$

Proof: (3.36) can be derived from (3.24) and (3.26). We show (3.37). Let $q = |p|^2 + t^2$. We fix μ and write x and p for x_μ and p_μ for notational simplicity in this proof. Then $[x, q] = 2ip$ and $x^2q = qx^2 + 2i(px + xp)$. We extend (3.25). From this we have

$$\frac{1}{q}x^2 = x^2\frac{1}{q} + 2i\frac{1}{q}(px + xp)\frac{1}{q} = x^2\frac{1}{q} + 2i\left(2\frac{1}{q}xp\frac{1}{q} - \frac{i}{q^2}\right).$$

Directly we can see that

$$2\frac{1}{q}xp\frac{1}{q} - \frac{i}{q^2} = x^2\frac{1}{q} + 4ix\frac{p}{q^2} - 8\frac{p^2}{q^3} + \frac{2}{q^2}.$$

We set $f = -8\frac{p^2}{q^3} + \frac{2}{q^2}$. Hence we have $\frac{1}{q}x^2 = x^2\frac{1}{q} + 4ix\frac{p}{q^2} + f$ and then

$$\frac{1}{q}x^4\frac{1}{q} = x^2\frac{1}{q^2}x^2 + 4i\left(x\frac{p}{q^3}x^2 - x^2\frac{p}{q^3}x\right) + (x^2\frac{f}{q} + \frac{f}{q}x^2) + x\frac{16p^2}{q^4}x + 4i(x\frac{pf}{q^2} - \frac{pf}{q^2}x) + f^2.$$

By a similar argument as the proof of the boundedness of Z mentioned in the proof of Lemma 3.10, we can get the desired results. \blacksquare

Lemma 3.12 *Suppose Assumption 2.1. Let $\Psi \in D(H_m^R)$. Then for $f \in C_0^\infty(\mathbb{R}^3 \setminus K)$,*

$$\int_{\mathbb{R}^3} dk f(k) (\Psi, e^{-it(H_m^R - E_m + \omega(k))} C_j(k) \langle x \rangle^2 \Phi_m) \in L^1([0, \infty), dt).$$

Proof: Let $1 \leq \mu \leq 3$ be fixed. We note that

$$e^{-is\omega} = \frac{i}{s} \frac{k_\mu}{\omega(k)} \nabla_\mu e^{-is\omega}, \quad e^{-is\omega} = -\frac{1}{s^2} \frac{k_\mu}{\omega(k)} \nabla_\mu \frac{k_\mu}{\omega(k)} \nabla_\mu e^{-is\omega}.$$

Since $C_j(k) = \frac{4}{\pi} I_j(k) \phi_\omega(k) + \rho_j(k) \frac{1}{\langle x \rangle^2}$, the integral is divided as

$$\begin{aligned} & \int_{\mathbb{R}^3} dk f(k) (\Psi, e^{-it(H_m^R - E_m + \omega(k))} C_j(k) \langle x \rangle^2 \Phi_m) \\ &= \int_{\mathbb{R}^3} dk f(k) (\Psi, e^{-it(H_m^R - E_m + \omega(k))} \frac{4}{\pi} I_j(k) \phi_\omega(k) \langle x \rangle^2 \Phi_m) \end{aligned} \quad (3.38)$$

$$+ \int_{\mathbb{R}^3} dk f(k) (\Psi, e^{-it(H_m^R - E_m + \omega(k))} \rho_j(k) \Phi_m). \quad (3.39)$$

We estimate (3.39). Integral by parts formula yields that

$$(3.39) = -\frac{1}{t^2} \int_{\mathbb{R}^3} dk e^{-it\omega(k)} (e^{it(H_m^R - E_m)} \Psi, \nabla_\mu \frac{k_\mu}{\omega(k)} \nabla_\mu \frac{k_\mu}{\omega(k)} f(k) \rho_j(k) \Phi_m)$$

and $(e^{-it(H_m^R - E_m)}\Psi, \nabla_\mu \frac{k_\mu}{\omega(k)} \nabla_\mu \frac{k_\mu}{\omega(k)} f(k) \rho_j(k) \Phi_m)$ is integrable. Hence $(3.39) \in L^1([0, \infty), dt)$. We now estimate (3.38). Then integral by parts formula also yields that

$$\begin{aligned} & \int_{\mathbb{R}^3} dk e^{-it\omega(k)} f(k) (e^{it(H_m^R - E_m)}\Psi, \frac{4}{\pi} I_j(k) \phi_\omega(k) \langle x \rangle^2 \Phi_m) \\ &= -\frac{i}{t} \int_{\mathbb{R}^3} dk e^{-it\omega(k)} \nabla_\mu \left(\frac{k_\mu}{\omega(k)} f(k) (e^{it(H_m^R - E_m)}\Psi, \frac{4}{\pi} I_j(k) \phi_\omega(k) \langle x \rangle^2 \Phi_m) \right). \end{aligned}$$

We shall see that

$$\begin{aligned} & \nabla_\mu \left(\frac{k_\mu}{\omega(k)} f(k) (e^{it(H_m^R - E_m)}\Psi, \frac{4}{\pi} I_j(k) \phi_\omega(k) \langle x \rangle^2 \Phi_m) \right) \\ &= (e^{it(H_m^R - E_m)}\Psi, \nabla_\mu \left(\frac{k_\mu}{\omega(k)} f(k) \frac{4}{\pi} I_j(k) \phi_\omega(k) \right) \langle x \rangle^2 \Phi_m) \end{aligned}$$

is integrable with respect to k . In order to see it we estimate

$$\nabla_\mu \frac{k_\mu}{\omega(k)} f(k) \frac{4}{\pi} I_j(k) \phi_\omega(k) = \text{I} + \text{II} + \text{III},$$

where

$$\begin{aligned} \text{I} &= \frac{4}{\pi} (\nabla_\mu \frac{k_\mu}{\omega(k)} f(k) \phi_\omega(k)) \int_0^\infty ds s^2 R_{s^2} T_j(k) (e^{-ikx} - 1) R_{s^2} \frac{1}{\langle x \rangle^2}, \\ \text{II} &= \frac{4}{\pi} \frac{k_\mu}{\omega(k)} f(k) \phi_\omega(k) \int_0^\infty ds s^2 R_{s^2} (\nabla_\mu e(k, j)) \cdot (p - A^R) (e^{-ikx} - 1) R_{s^2} \frac{1}{\langle x \rangle^2}, \\ \text{III} &= \frac{4}{\pi} \frac{k_\mu}{\omega(k)} f(k) \phi_\omega(k) \int_0^\infty ds s^2 R_{s^2} T_j(k) (-ix_\mu) e^{-ikx} R_{s^2} \frac{1}{\langle x \rangle^2}. \end{aligned}$$

We can estimate I and II in a similar manner to the proof of Lemma 3.10. Hence I and II are bounded with

$$\|\text{I} + \text{II}\| \leq C \left| \nabla_\mu \frac{k_\mu}{\omega(k)} f(k) + \frac{k_\mu}{\omega(k)} f(k) \right| (|k| + |k|^2) \quad (3.40)$$

with some constant C independent of t . Let us investigate III. We have $\text{III} = \text{III}_1 + \text{III}_2$, where

$$\begin{aligned} \text{III}_1 &= \frac{k_\mu}{\omega(k)} f(k) \int_0^\infty ds s^2 R_{s^2} T_j(k) e^{-ikx} R_{s^2} \frac{-ix_\mu}{\langle x \rangle^2}, \\ \text{III}_2 &= \frac{k_\mu}{\omega(k)} f(k) \int_0^\infty ds s^2 R_{s^2} T_j(k) e^{-ikx} [-ix_\mu, R_{s^2}] \frac{1}{\langle x \rangle^2}. \end{aligned}$$

In a similar way to the proof of Lemma 3.10 again, III_1 can be also estimated as

$$\|\text{III}_1\| \leq C \left| \frac{k_\mu}{\omega(k)} f(k) \right| (|k| + |k|^2). \quad (3.41)$$

We estimate III₂. We have

$$\text{III}_2 = \frac{k_\mu}{\omega(k)} f(k) \int_0^\infty ds s^2 R_{s^2} T_j(k) e^{-ikx} R_{s^2} (p_\mu - A_{R\mu}) R_{s^2} \frac{1}{\langle x \rangle^2}$$

and we divide the integral as $L_1 + L_2$ where $L_1 = \int_0^1 ds \dots$ and $L_2 = \int_1^\infty ds \dots$. We can see that

$$\|L_2 \Psi\| \leq \int_1^\infty ds s \|R_{s^2} T_j(k)\| s \|R_{s^2} (p_\mu - A_{R\mu})\| \|R_{s^2} \frac{1}{\langle x \rangle^2} \Psi\| \leq C \int_1^\infty ds \frac{1}{s^2} \|\Psi\|,$$

where we used that $s \|R_{s^2} T_j(k)\| \leq s \|\sqrt{T_p} R_{s^2}\| \leq \frac{1}{2}$ and $s \|R_{s^2} (p_\mu - A_{R\mu})\| \leq s \|\sqrt{T_p} R_{s^2}\| \leq \frac{1}{2}$. We also see that

$$\begin{aligned} \|L_1 \Psi\| &\leq \left(\int_0^1 ds s \|R_{s^2} T_j(k)\|^2 \right)^{\frac{1}{2}} \left(\int_0^1 ds s \|R_{s^2} (p_\mu - A_{R\mu})\|^2 s^2 \|R_{s^2} \frac{1}{\langle x \rangle^2} \Psi\|^2 \right)^{\frac{1}{2}} \\ &\leq \left(\int_0^1 ds s \|R_{s^2} T_j(k)\|^2 \right)^{\frac{1}{2}} \left(\int_0^1 ds s \|R_{s^2} (p_\mu - A_{R\mu})\|^2 \right)^{\frac{1}{2}} \|\Psi\| \\ &\leq \int_0^1 ds s \|\sqrt{T_p} R_{s^2}\|^2 \|\Psi\| \leq C \|\Psi\|. \end{aligned}$$

Thus III₂ is also bounded with

$$\|\text{III}_2\| \leq C \left| \frac{k_\mu}{\omega(k)} f(k) \right|. \quad (3.42)$$

Then we can conclude that

$$\begin{aligned} &|(e^{it(H_m^R - E_m)} \Psi, \nabla_\mu \left(\frac{k_\mu}{\omega(k)} f(k) C_j(k) \right) \langle x \rangle^2 \Phi_m)| \\ &\leq C \|\Psi\| \|\langle x \rangle^2 \Phi_m\| \left| \nabla_\mu \frac{k_\mu}{\omega(k)} f(k) + \frac{k_\mu}{\omega(k)} f(k) \right| (1 + |k| + |k|^2), \end{aligned}$$

and hence

$$\int_{\mathbb{R}^3} dk \int_0^\infty ds s^2 |(e^{it(H_m^R - E_m)} \Psi, R_{s^2} A_k R_{s^2} \Phi_m)| < \infty,$$

where

$$\begin{aligned} A_k &= \nabla_\mu \left(\frac{k_\mu}{\omega(k)} f(k) \right) T_j(k) (e^{-ikx} - 1) + \frac{k_\mu}{\omega(k)} f(k) (\nabla_\mu e(k, j)) (p - A_R) (e^{-ikx} - 1) \\ &\quad + \frac{k_\mu}{\omega(k)} T_j(k) (-ix_\mu) e^{-ikx}. \end{aligned}$$

By Fubini's lemma we can exchange integrals $\int dk$ and $\int ds$ and we see that

$$\begin{aligned} & \int_{\mathbb{R}^3} dk e^{-it\omega} \nabla_\mu \left(\frac{k_\mu}{\omega(k)} f(k) (e^{it(H_m^R - E_m)} \Psi, \mathbf{I}_j(k) \phi_\omega(k) \langle x \rangle^2 \Phi_m) \right) \\ &= \int_0^\infty ds s^2 (R_{s^2} e^{it(H_m^R - E_m)} \Psi, (\xi_t - ix_\mu \xi_t(x)) (p - A_R) R_{s^2} \Phi_m), \end{aligned}$$

where

$$\begin{aligned} \xi_t(x) &= \int_{\mathbb{R}^3} dk e^{-it\omega(k)} \nabla_{k_\mu} \left(\frac{k_\mu}{\omega(k)} f(k) \phi_\omega(k) e^{-ikx} e(k, j) \right), \\ \xi_t &= \xi_t(0) = \int_{\mathbb{R}^3} dk e^{-it\omega(k)} \nabla_{k_\mu} \left(\frac{k_\mu}{\omega(k)} f(k) \phi_\omega(k) e(k, j) \right). \end{aligned}$$

Inserting

$$\begin{aligned} & \nabla_{k_\mu} \frac{k_\mu}{\omega(k)} f(k) \phi_\omega(k) e(k, j) e^{-ikx} \\ &= \nabla_{k_\mu} \left(\frac{k_\mu}{\omega(k)} f(k) \phi_\omega(k) e(k, j) \right) e^{-ikx} - ix_\mu \frac{k_\mu}{\omega(k)} f(k) \phi_\omega(k) e(k, j) e^{-ikx} \end{aligned}$$

into $\xi_t(x)$, we then see that $\xi_t(x) = \xi_t^{(1)}(x) - ix_\mu \xi_t^{(2)}(x)$, where

$$\begin{aligned} \xi_{t,\nu}^{(1)}(x) &= \int_{\mathbb{R}^3} dk e^{-it\omega(k) - ikx} \nabla_{k_\mu} \left(\frac{k_\mu}{\omega(k)} f(k) \phi_\omega(k) e_\mu(k, j) \right), \\ \xi_{t,\nu}^{(2)}(x) &= \int_{\mathbb{R}^3} dk e^{-it\omega(k) - ikx} \frac{k_\mu}{\omega(k)} f(k) \phi_\omega(k) e_\nu(k, j). \end{aligned}$$

Since $\frac{k_\mu}{\omega(k)} f(k) \phi_\omega(k) e_\nu(k, j) \in C_0^\infty(\mathbb{R}_k^3 \setminus \{0\})$ for $\mu = 1, 2, 3$ by the assumption on $\hat{\varphi}$ and f . We also note that

$$\sup_{x \in \mathbb{R}^3} |\xi_t^{(j)}(x)| \leq \frac{C}{1+t} \quad (3.43)$$

for $j = 1, 2$. Refer to see [RS79, Theorem XI.19(c)] for (3.43). Since

$$\begin{aligned} & |(R_{s^2} e^{it(H_m^R - E_m)} \Psi, (\xi_t + \xi_t^{(1)}(x)) (p - A_R) R_{s^2} \Phi_m)| \\ &= |(R_{s^2} e^{it(H_m^R - E_m)} \Psi, (p - A_R) (\xi_t + \xi_t^{(1)}(x)) R_{s^2} \Phi_m)| \leq \frac{C}{t+1} \|\sqrt{T_p} R_{s^2} e^{it(H_m^R - E_m)} \Psi\| \|R_{s^2} \Phi_m\|, \end{aligned}$$

we have

$$\begin{aligned} & \int_0^1 ds s^2 |(R_{s^2} e^{it(H_m^R - E_m)} \Psi, (\xi_t + \xi_t^{(1)}(x)) (p - A_R) R_{s^2} \Phi_m)| \\ & \leq \frac{C}{t+1} \left(\int_0^1 ds s |\sqrt{T_p} R_{s^2} e^{it(H_m^R - E_m)} \Psi|^2 \right)^{1/2} \left(\int_0^1 ds s^3 \|R_{s^2} \Phi_m\|^2 \right)^{1/2}. \end{aligned}$$

We already see that $\int_0^1 ds s \|\sqrt{T_p} R_{s^2} e^{it(H_m^R - E_m)} \Psi\|^2$ is finite in (3.23), and moreover

$$\int_0^1 ds s^3 \|R_{s^2} \Phi_m\|^2 \leq \frac{1}{2} \left\| \frac{1}{|p|} \Phi_m \right\|^2 \leq \frac{1}{2} \| |x| \Phi_m \|^2 < \infty$$

by the Hardy-Rellich inequality. Similarly we can see that

$$\begin{aligned} & \int_0^1 ds s^2 |(R_{s^2} e^{it(H_m - E_m)} \Psi, -ix_\mu \xi_t^{(2)}(x))(p - A_R) R_{s^2} \Phi_m| \\ & \leq \frac{C}{1+t} \left(\int_0^1 ds s |(\sqrt{T_p} R_{s^2} e^{it(H_m - E_m)} \Psi)|^2 \right)^{1/2} \left(\int_0^1 ds s^3 \| |x| R_{s^2} \Phi_m \|^2 \right)^{1/2} < \infty. \end{aligned}$$

Next we can estimate $\int_1^\infty \dots ds$. we have

$$\begin{aligned} & \int_1^\infty ds s^2 |(R_{s^2} e^{it(H_m^R - E_m)} \Psi, (\xi_t + \xi_t^{(1)}(x))(p - A_R) R_{s^2} \Phi_m)| \\ & \leq \frac{C}{t+1} \int_1^\infty ds s^2 \|R_{s^2} e^{it(H_m^R - E_m)} \Psi\| \|\sqrt{T_p} R_{s^2} \Phi_m\| \\ & \leq \frac{C}{t+1} \int_1^\infty ds \frac{1}{s^2} \|\Psi\| \|(H_m + \mathbb{1}) \Phi_m\| < \infty. \end{aligned}$$

In order to estimate $-ix_\mu \xi_t^{(2)}(p - A_R) R_{s^2}$ we compute the commutation relation:

$$\begin{aligned} & -ix_\mu \xi_t^{(2)}(p - A_R) R_{s^2} \\ & = \xi_t^{(2)}(x)(p - A_R) R_{s^2} (-ix_\mu) + 2\xi_t^{(2)}(x)(p - A_R) R_{s^2} (p_\mu - A_{R\mu}) R_{s^2} + \xi_{t,\mu}^{(2)}(x) R_{s^2} \end{aligned}$$

and then

$$\begin{aligned} & \int_1^\infty ds s^2 |(R_{s^2} e^{it(H_m^R - E_m)} \Psi, \xi_t^{(2)}(x)(p - A_R) R_{s^2} (-ix_\mu) \Phi_m)| \\ & \leq \frac{C}{t+1} \int_1^\infty ds s^2 \|\sqrt{T_p} R_{s^2} e^{it(H_m^R - E_m)} \Psi\| \|R_{s^2} (-ix_\mu) \Phi_m\| \\ & \leq \frac{C}{t+1} \int_1^\infty ds \frac{1}{s^2} \|(H_m + \mathbb{1}) \Psi\| \| |x| \Phi_m \| < \infty. \end{aligned}$$

Estimates of the remaining terms are straightforward:

$$\begin{aligned} (1) \quad & \int_1^\infty ds s^2 |(R_{s^2} e^{it(H_m^R - E_m)} \Psi, 2\xi_t^{(2)}(x)(p - A_R) R_{s^2} (p_\mu - A_{R\mu}) R_{s^2} \Phi_m)| \\ & \leq \frac{C}{t+1} \int_1^\infty ds \frac{1}{s^4} \|\Psi\| \|\Phi_m\| < \infty, \\ (2) \quad & \int_1^\infty ds s^2 |(R_{s^2} e^{it(H_m^R - E_m)} \Psi, \xi_{t,\mu}^{(2)}(x) R_{s^2} \Phi_m)| \leq \frac{C}{t+1} \int_1^\infty ds \frac{1}{s^2} \|\Psi\| \|\Phi_m\| < \infty. \end{aligned}$$

Hence

$$\left| \int_{\mathbb{R}^3} dk f(k) (\Psi, e^{-it(H_m^R - E_m + \omega(k))} C_j(k) \langle x \rangle^2 \Phi_m) \right| \leq \frac{C}{t(t+1)}$$

and the left-hand side above is integrable with respect to t . ■

Lemma 3.13 *Suppose Assumptions 2.1. Then $\Phi_m \in D(N^{\frac{1}{2}})$ if and only if*

$$\int_{\mathbb{R}^3} \|(H_m^R - E_m + \omega(k))^{-1} C_j(k) \langle x \rangle^2 \Phi_m\|^2 dk < \infty. \quad (3.44)$$

Furthermore if $\Phi_m \in D(N^{\frac{1}{2}})$, then the identity

$$\|N^{\frac{1}{2}} \Phi_m\|^2 = \int_{\mathbb{R}^3} \|(H_m^R - E_m + \omega(k))^{-1} C_j(k) \langle x \rangle^2 \Phi_m\|^2 dk. \quad (3.45)$$

follows.

Proof: By the general proposition, Proposition 3.7, the proof can be proven under the identifications: $C_j(k) \langle x \rangle^2$ and $B_j(k)$ in Proposition 3.7, by Lemmas 3.9 and 3.12. ■

Corollary 3.14 *Suppose Assumptions 2.1 and*

$$\alpha = \int_{\mathbb{R}^3} \left(\frac{|k| + |k|^2}{\omega(k)} \right)^2 \frac{\hat{\varphi}(k)^2}{\omega(k)} dk < \infty. \quad (3.46)$$

Then $\Phi_m \in D(N^{\frac{1}{2}})$ and

$$\|N^{\frac{1}{2}} \Phi_m\|^2 \leq C \alpha \|\langle x \rangle^2 \Phi_m\|^2 \quad (3.47)$$

with some constant C independent of m .

Proof: By the assumption we can check (3.44). Then the corollary follows from Lemma 3.13. ■

3.3 Weak derivative of Φ_m

3.3.1 Extended Hilbert space

In this section we shall derive the weak derivative of Φ_m . Throughout this section we assume that $m > 0$. We write as $\Phi_m = \{\Phi_m^{(n)}\}_{n=0}^\infty \in \mathcal{H} = \bigoplus_{n=0}^\infty \mathcal{H}^{(n)}$ and we shall show that $\Phi_m^{(n)} \in W^{1,p}(\Omega)$, i.e., $\nabla \Phi_m^{(n)} \in L^p(\Omega)$ for $1 \leq p < 2$ and $n \geq 1$ with any bounded domain $\Omega \subset \mathbb{R}_x^3 \times \mathbb{R}_k^{3n}$. Note that $\Phi_m^{(0)} \in \mathbb{C}$ and $\nabla \Phi_m^{(0)} = 0$. The idea is to apply Corollary 3.15.

Corollary 3.15 *Suppose Assumptions 2.1 and $f, f/\sqrt{\omega} \in L^2(\mathbb{R}^3)$. Then*

$$a(f, j) \Phi_m = - \int_{\mathbb{R}^3} f(k) (H_m^R - E_m + \omega(k))^{-1} C_j(k) \langle x \rangle^2 \Phi_m dk.$$

Proof: This follows from Proposition 3.8. ■

Let

$$R(k) = (H_m^R - E_m + \omega(k))^{-1}.$$

We define

$$X = \int_{\mathbb{R}^3}^{\oplus} R(k) C_j(k) \langle x \rangle^2 \Phi_m dk \in \int_{\mathbb{R}^3}^{\oplus} \mathcal{H} dk \cong L^2(\mathbb{R}_k^3) \otimes \mathcal{H},$$

and

$$X^{n+1} = \int_{\mathbb{R}^3}^{\oplus} (R(k) C_j(k) \langle x \rangle^2 \Phi_m)^{(n)} dk \in \int_{\mathbb{R}^3}^{\oplus} \mathcal{H}^{(n)} dk \cong L^2(\mathbb{R}_k^3) \otimes \mathcal{H}^{(n)}.$$

Lemma 3.16 *Suppose Assumptions 2.1, $f, f/\sqrt{\omega} \in L^2(\mathbb{R}_k^3)$ and $G \in \mathcal{H}^{(n)}$. Then we have the identity*

$$(\bar{f} \otimes G, \Phi_m^{(n+1)})_{\mathcal{H}^{(n+1)}} = -\frac{1}{\sqrt{n+1}} (\bar{f} \otimes G, X^{n+1})_{\mathcal{H}^{(n+1)}}, \quad (3.48)$$

where we use the identification:

$$\mathcal{H}^{(n+1)} \cong L^2(\mathbb{R}_k^3) \otimes \mathcal{H}^{(n)} \cong \int_{\mathbb{R}^3}^{\oplus} \mathcal{H}^{(n)} dk.$$

Proof: We have from Corollary 3.15 that

$$(\Psi, a(f, j) \Phi_m)_{\mathcal{H}} = - \left(\Psi, \int_{\mathbb{R}^3} dk f(k) R(k) C_j(k) \langle x \rangle^2 \Phi_m \right)_{\mathcal{H}} \quad (3.49)$$

for any $\Psi \in \mathcal{H}$. Taking $\Psi = (0, \dots, 0, \overset{n_{th}}{G}, 0, \dots) \in \mathcal{H}$, where $G \in \mathcal{H}^{(n)}$. Since

$$\begin{aligned} (\Psi, a(f, j) \Phi_m^{(n+1)})_{\mathcal{H}^{(n)}} &= (a^\dagger(\bar{f}, j) \Psi, \Phi_m^{(n+1)})_{\mathcal{H}^{(n+1)}} = \sqrt{n+1} (S_{n+1}(\bar{f} \otimes G), \Phi_m^{(n+1)})_{\mathcal{H}^{(n+1)}} \\ &= \sqrt{n+1} (\bar{f} \otimes G, S_{n+1} \Phi_m^{(n+1)})_{\mathcal{H}^{(n+1)}} = \sqrt{n+1} (\bar{f} \otimes G, \Phi_m^{(n+1)})_{\mathcal{H}^{(n+1)}}, \end{aligned} \quad (3.50)$$

where S_{n+1} denotes the symmetrizer. On the other hand we can see that

$$\begin{aligned} \left(\Psi, \int_{\mathbb{R}^3} dk f(k) R(k) C_j(k) \langle x \rangle^2 \Phi_m \right)_{\mathcal{H}} &= \int_{\mathbb{R}^3} dk f(k) (\Psi, R(k) C_j(k) \langle x \rangle^2 \Phi_m)_{\mathcal{H}} \\ &= \int_{\mathbb{R}^3} dk f(k) (G, (R(k) C_j(k) \langle x \rangle^2 \Phi_m)^{(n)})_{\mathcal{H}^{(n)}} = (\bar{f} \otimes G, X^{n+1})_{\mathcal{H}^{(n+1)}}. \end{aligned} \quad (3.51)$$

By (3.50) and (3.51) the lemma follows. ■

Let us consider the weak derivative of $\Phi_m^{(n+1)}$. Let $\varepsilon_1 = (\varepsilon, 0, 0)$, $\varepsilon_2 = (0, \varepsilon, 0)$ and $\varepsilon_3 = (0, 0, \varepsilon)$. We can see that

$$(\nabla_{k_\mu} f \otimes G, \Phi)_{\mathcal{H}^{(n+1)}} = \frac{1}{\sqrt{n+1}} \lim_{\varepsilon \rightarrow 0} (f \otimes G, X_{\varepsilon_\mu}^{n+1})_{\mathcal{H}^{(n+1)}}, \quad (3.52)$$

where $X_{\varepsilon_\mu}^{n+1} = \int_{\mathbb{R}^3}^{\oplus} X_{\varepsilon_\mu}^n(k) dk$ with

$$X_{\varepsilon_\mu}^n(k) = \left(\frac{R(k + \varepsilon_\mu) C_j(k + \varepsilon_\mu) - R(k) C_j(k)}{\varepsilon} \langle x \rangle^2 \Phi_m \right)^{(n)}. \quad (3.53)$$

In the next section we investigate the convergence of sequence $\{X_{\varepsilon_\mu}^{n+1}\}$ as $\varepsilon \rightarrow 0$.

3.3.2 Uniform continuity

We generalize (3.53). For each $k \in \mathbb{R}^3$ and $h \in \mathbb{R}^3$ we define

$$X_h(k) = \frac{R(k+h)C_j(k+h) - R(k)C_j(k)}{|h|} \langle x \rangle^2 \Phi_m$$

$$X_h^n(k) = \left(\frac{R(k+h)C_j(k+h) - R(k)C_j(k)}{|h|} \langle x \rangle^2 \Phi_m \right)^{(n)} \in \mathcal{H}_n$$

and

$$X_h = \int_{\mathbb{R}^3}^{\oplus} X_h(k) dk \in \int_{\mathbb{R}^3}^{\oplus} \mathcal{H} dk,$$

$$X_h^{n+1} = \int_{\mathbb{R}^3}^{\oplus} X_h^n(k) dk \in \int_{\mathbb{R}^3}^{\oplus} \mathcal{H}_n dk.$$

Let us consider $X_h(k)$ for each $k \in \mathbb{R}^3 \setminus K$ and we divide $X_h(k)$ as

$$X_h(k) = \frac{R(k+h) - R(k)}{|h|} C_j(k) \langle x \rangle^2 \Phi_m + R(k+h) \frac{C_j(k+h) - C_j(k)}{|h|} \langle x \rangle^2 \Phi_m$$

for each $k \in \mathbb{R}^3 \setminus K$ and $h \in \mathbb{R}^3$. Suppose that $2|h_\mu| \leq |k_\mu|$ for $\mu = 1, 2, 3$. Then we have $\omega(k+h) \geq \frac{1}{2}\omega(k)$. This bound is used often times in lemmas below.

Lemma 3.17 *Suppose that $2|h_\mu| \leq |k_\mu|$ for $\mu = 1, 2, 3$, and $\text{supp } \hat{\varphi} \subset \{k \in \mathbb{R}^3 \mid |k| \leq 2\Lambda\}$ for some Λ . Then it follows that for each $k \in \mathbb{R}^3 \setminus K$,*

$$\left\| \frac{R(k+h) - R(k)}{|h|} C_j(k) \langle x \rangle^2 \Phi_m \right\|_{\mathcal{H}} \leq \frac{C \mathbb{1}_{|k| \leq \Lambda} (1 + |k|)}{\sqrt{\omega(k)} \sqrt{k_1^2 + k_2^2}} \|\langle x \rangle^2 \Phi_m\|_{\mathcal{H}} \quad j = 1, 2, \quad (3.54)$$

where $\mathbb{1}_{|k| \leq \Lambda}$ is the characteristic function of $\{k \in \mathbb{R}^3 \mid |k| \leq 2\Lambda\}$.

Proof: We see that

$$\left| \frac{\omega(k+h) - \omega(k)}{|h|} \right| \leq 1 \leq \frac{\omega(k)}{\sqrt{k_1^2 + k_2^2}}. \quad (3.55)$$

Then

$$\left\| \frac{R(k+h) - R(k)}{|h|} \right\| = \left| \frac{\omega(k+h) - \omega(k)}{|h|} \right| \|R(k+h)R(k)\| \leq \frac{2}{\sqrt{k_1^2 + k_2^2} \omega(k)}. \quad (3.56)$$

Since $\|C_j(K)\| \leq C(|k| + |k|^2)|\phi_\omega(k)|$ and $\frac{|k|}{\omega(k)} < 1$, (3.54) follows. \blacksquare

Lemma 3.18 *Suppose that $2|h_\mu| \leq |k_\mu|$ for $\mu = 1, 2, 3$, and $\text{supp } \hat{\varphi} \subset \{k \in \mathbb{R}^3 \mid |k| \leq 2\Lambda\}$ for some Λ . Then it follows that for each $k \in \mathbb{R}^3 \setminus K$,*

$$\left\| R(k+h) \frac{C_j(k+h) - C_j(k)}{|h|} \langle x \rangle^2 \Phi_m \right\|_{\mathcal{H}} \leq \frac{C \mathbb{1}_{|k| \leq \Lambda} (1 + |k|)}{\sqrt{\omega(k)} \sqrt{k_1^2 + k_2^2}} \|\langle x \rangle^2 \Phi_m\|_{\mathcal{H}}. \quad (3.57)$$

Proof: By the definition of $C_j(k)$ we have

$$\begin{aligned} R(k+h) \frac{C_j(k+h) - C_j(k)}{|h|} \langle x \rangle^2 \Phi_m &= R(k+h) \frac{\rho_j(k+h) - \rho_j(k)}{|h|} \Phi_m \\ &+ R(k+h) \frac{4}{\pi} (\phi_\omega(k+h) \mathbb{I}_j(k+h) - \phi_\omega(k) \mathbb{I}_j(k)) \langle x \rangle^2 \Phi_m, \end{aligned} \quad (3.58)$$

Let us estimate the first term of the right-hand side of (3.58). Note that for $j = 1, 2$, $|\nabla \cdot e(k, j)| \leq \frac{C}{\sqrt{k_1^2 + k_2^2}}$, and that

$$\left| \frac{e(k+h, j) - e(k, j)}{|h|} \right| \leq |\nabla \cdot e(k + \theta h, j)| \leq \frac{C}{\sqrt{k_1^2 + k_2^2}}. \quad (3.59)$$

See Appendix B. Furthermore it is straightforward to see that

$$\frac{1}{|h|} |\phi_\omega(k+h) - \phi_\omega(k)| \leq C \left(\frac{1}{\sqrt{\omega}} + \frac{1}{\omega^{3/2}} \right) \leq \frac{C \mathbb{1}_{|k| \leq \Lambda}}{\sqrt{\omega(k)} \sqrt{k_1^2 + k_2^2}}. \quad (3.60)$$

Then we obtain that

$$\left| \frac{\rho(k+h) - \rho(k)}{|h|} \right| \leq \frac{C \omega(k) \mathbb{1}_{|k| \leq \Lambda} |x|}{\sqrt{\omega(k)} \sqrt{k_1^2 + k_2^2}}. \quad (3.61)$$

Thus

$$\left\| R(k+h) \frac{\rho_j(k+h) - \rho_j(k)}{h} \Phi_m \right\| \leq \frac{C \mathbb{1}_{|k| \leq \Lambda}}{\sqrt{\omega(k)} \sqrt{k_1^2 + k_2^2}} \|\langle x \rangle^2 \Phi_m\| \quad (3.62)$$

follows. Next we shall show that

$$\left\| R(k+h) (\phi_\omega(k+h) \mathbb{I}_j(k+h) - \phi_\omega(k) \mathbb{I}_j(k)) \langle x \rangle^2 \Phi_m \right\| \leq \frac{C \mathbb{1}_{|k| \leq \Lambda} (1 + |k|)}{\sqrt{\omega(k)} \sqrt{k_1^2 + k_2^2}} \|\langle x \rangle^2 \Phi_m\|. \quad (3.63)$$

We have

$$\begin{aligned} &R(k+h) (\phi_\omega(k+h) \mathbb{I}_j(k+h) - \phi_\omega(k) \mathbb{I}_j(k)) \langle x \rangle^2 \Phi_m \\ &= R(k+h) \frac{\mathbb{I}_j(k+h) - \mathbb{I}_j(k)}{|h|} \phi_\omega(k+h) \langle x \rangle^2 \Phi_m + R(k+h) \mathbb{I}_j(k) \frac{1}{|h|} (\phi_\omega(k+h) - \phi_\omega(k)) \langle x \rangle^2 \Phi_m. \end{aligned} \quad (3.64)$$

The second term of the right-hand side of (3.64) can be estimated as

$$\begin{aligned} &\left\| R(k+h) \left\| \mathbb{I}_j(k) \frac{1}{|h|} (\phi_\omega(k+h) - \phi_\omega(k)) \langle x \rangle^2 \Phi_m \right\| \right\| \\ &\leq \|R(k+h)\| \|\mathbb{I}_j(k)\| \frac{C \mathbb{1}_{|k| \leq \Lambda}}{\sqrt{\omega(k)} \sqrt{k_1^2 + k_2^2}} \|\langle x \rangle^2 \Phi_m\| \leq \frac{C(1 + |k|) \mathbb{1}_{|k| \leq \Lambda}}{\sqrt{\omega(k)} \sqrt{k_1^2 + k_2^2}} \|\langle x \rangle^2 \Phi_m\|. \end{aligned}$$

We can also show that

$$\left\| R(k+h) \frac{I_j(k+h) - I_j(k)}{|h|} \phi_\omega(k+h) \langle x \rangle^2 \Phi_m \right\| \leq \frac{C(1+|k|) \mathbb{1}_{|k| \leq \Lambda}}{\sqrt{\omega(k)} \sqrt{k_1^2 + k_2^2}} \|\langle x \rangle^2 \Phi_m\|. \quad (3.65)$$

(3.65) is proven by Lemmas 3.19 below. Hence the lemma follows. \blacksquare

Lemma 3.19 *Suppose that $2|h_\mu| \leq |k_\mu|$ for $\mu = 1, 2, 3$, and $\text{supp } \hat{\varphi} \subset \{k \in \mathbb{R}^3 \mid |k| \leq 2\Lambda\}$ for some Λ . Then (3.65) follows for each $k \in \mathbb{R}^3 \setminus K$.*

Proof: The proof is similar to show the boundedness of $I_j(k)$ which is given in Lemma 3.10. Set $I_j(k) = I_{1,j}(k) + I_{2,j}(k)$, where we recall that

$$\begin{aligned} I_{1,j}(k) &= \int_0^1 dt t^2 R_{t^2} T_j(k) (e^{-ikx} - 1) R_{t^2} \frac{1}{\langle x \rangle^2}, \\ I_{2,j}(k) &= \int_1^\infty dt t^2 R_{t^2} T_j(k) (e^{-ikx} - 1) R_{t^2} \frac{1}{\langle x \rangle^2}. \end{aligned}$$

We shall prove both bounds below:

$$\left\| \frac{I_{1,j}(k+h) - I_{1,j}(k)}{|h|} \right\| \leq \frac{C|k|}{\sqrt{k_1^2 + k_2^2}}, \quad (3.66)$$

$$\left\| \frac{I_{2,j}(k+h) - I_{2,j}(k)}{|h|} \right\| \leq C(1+|k|), \quad j = 1, 2. \quad (3.67)$$

We have

$$\frac{I_{1,j}(k+h) - I_{1,j}(k)}{|h|} = J_{1,j}^{(1)} + J_{1,j}^{(2)},$$

where

$$\begin{aligned} J_{1,j}^{(1)} &= \int_0^1 dt t^2 R_{t^2} \frac{T_j(k+h) - T_j(k)}{|h|} (e^{-ikx} - 1) R_{t^2} \frac{1}{\langle x \rangle^2}, \\ J_{1,j}^{(2)} &= \int_0^1 dt t^2 R_{t^2} T_j(k+h) \frac{e^{-i(k+h)x} - e^{-ikx}}{|h|} R_{t^2} \frac{1}{\langle x \rangle^2}. \end{aligned}$$

By (3.59), we see that

$$\left\| \frac{T_j(k+h) - T_j(k)}{|h|} R_{t^2} \Psi \right\| \leq \frac{C}{\sqrt{k_1^2 + k_2^2}} \|\sqrt{T_p} R_{t^2} \Psi\|.$$

Thus we obtain by (3.44) that

$$|(\Psi, J_{1,j}^{(1)} \Phi)| \leq C \frac{|k|}{\sqrt{k_1^2 + k_2^2}} \|\Psi\| \left(\int_0^1 dt t^3 \left\| |x| R_{t^2} \frac{1}{\langle x \rangle^2} \Phi \right\|^2 \right)^{\frac{1}{2}} \leq \frac{C|k|}{\sqrt{k_1^2 + k_2^2}} \|\Psi\| \|\Phi\|.$$

We also obtain that

$$|(\Psi, J_{1,j}^{(2)}\Phi)| \leq C\|\Psi\| \left(\int_0^1 dt t^3 \left\| |x| R_{t^2} \frac{1}{\langle x \rangle^2} \Phi \right\|^2 \right)^{\frac{1}{2}} \leq C\|\Psi\| \|\Phi\|.$$

Thus (3.66) follows. We have

$$\begin{aligned} \frac{I_{2,j}(k+h) - I_{2,j}(k)}{|h|} &= \int_1^\infty dt t^2 R_{t^2} \frac{T_j(k+h) - T_j(k)}{|h|} (e^{-ikx} - 1) R_{t^2} \frac{1}{\langle x \rangle^2} \\ &\quad + \int_1^\infty dt t^2 R_{t^2} T_j(k+h) \frac{e^{-i(k+h)x} - e^{-ikx}}{|h|} R_{t^2} \frac{1}{\langle x \rangle^2}. \end{aligned} \quad (3.68)$$

We consider the first term of the right-hand side of (3.68). Let us recall that $Y(k) = -2k \cdot (p - A_R)$, and set

$$\begin{aligned} J_2^{(1)} &= \int_1^\infty dt t^2 R_{t^2} \frac{T_j(k+h) - T_j(k)}{|h|} R_{t^2}^{(k)} \frac{e^{-ikx} - 1}{\langle x \rangle^2}, \\ J_2^{(2)} &= -2 \int_1^\infty dt t^2 R_{t^2} \frac{T_j(k+h) - T_j(k)}{|h|} R_{t^2} Y(k) R_{t^2}^{(k)} \frac{1}{\langle x \rangle^2}, \\ J_2^{(3)} &= -|k|^2 \int_1^\infty dt t^2 R_{t^2} \frac{T_j(k+h) - T_j(k)}{|h|} R_{t^2} R_{t^2}^{(k)} \frac{1}{\langle x \rangle^2}. \end{aligned}$$

Then

$$\int_1^\infty dt t^2 R_{t^2} \frac{T_j(k+h) - T_j(k)}{|h|} (e^{-ikx} - 1) R_{t^2} \frac{1}{\langle x \rangle^2} = J_2^{(1)} + J_2^{(2)} + J_2^{(3)}.$$

Note that

$$\left| \left(\Psi, \frac{T_j(k+h) - T_j(k)}{|h|} \Phi \right) \right| \leq \frac{C}{\sqrt{k_1^2 + k_2^2}} \|T_p^{\frac{1}{4}} \Psi\| \|T_p^{\frac{1}{4}} \Phi\|.$$

Then it can be estimated as

$$\left\| \int_1^\infty dt t^2 R_{t^2} \frac{T_j(k+h) - T_j(k)}{|h|} (e^{-ikx} - 1) R_{t^2} \frac{1}{\langle x \rangle^2} \right\| \leq \|J_2^{(1)}\| + \|J_2^{(2)}\| + \|J_2^{(3)}\| \leq \frac{C(|k| + |k|^2)}{\sqrt{k_1^2 + k_2^2}}. \quad (3.69)$$

Next we consider the second term of the right-hand side of (3.68). Set

$$\begin{aligned} K_2^{(1)} &= \int_1^\infty dt t^2 R_{t^2} T_j(k) e^{-ikx} R_{t^2}^{(h)} \frac{e^{-ihx} - 1}{|h|} \frac{1}{\langle x \rangle^2}, \\ K_2^{(2)} &= -2 \int_1^\infty dt t^2 R_{t^2} T_j(k) e^{-ikx} R_{t^2} \frac{Y(h)}{|h|} R_{t^2}^{(h)} \frac{1}{\langle x \rangle^2}, \\ K_2^{(3)} &= -|h| \int_1^\infty dt t^2 R_{t^2} T_j(k) e^{-ikx} R_{t^2} R_{t^2} \frac{1}{\langle x \rangle^2}. \end{aligned}$$

Then

$$\int_1^\infty dt t^2 R_{t^2} T_j(k+h) \frac{e^{-i(k+h)x} - e^{-ikx}}{|h|} R_{t^2} \frac{1}{\langle x \rangle^2} = K_2^{(1)} + K_2^{(2)} + K_2^{(3)}.$$

For all $\Psi, \Phi \in \mathcal{H}$, we have

$$|(\Psi, K_2^{(1)} \Phi)| \leq \int_1^\infty dt t^2 \|T_p^{\frac{1}{4}} R_{t^2} \Psi\| \|T_p^{\frac{1}{4}} e^{-ikx} R_{t^2}^{(h)} \tilde{\Phi}\|,$$

where $\tilde{\Phi} = \frac{e^{-ihx}-1}{|h|} \frac{1}{\langle x \rangle^2} \Phi$. Note that

$$\begin{aligned} \|T_p^{\frac{1}{4}} e^{-ikx} R_{t^2}^{(h)} \tilde{\Phi}\| &= \|e^{-ikx} |(p+h-A_R-(h+k))^2|^{\frac{1}{4}} R_{t^2}^{(h)} \tilde{\Phi}\| \\ &\leq \|T_{p+h}^{\frac{1}{4}} R_{t^2}^{(h)} \tilde{\Phi}\| + (\sqrt{|h|} + \sqrt{|k|}) \|R_{t^2}^{(h)} \tilde{\Phi}\|. \end{aligned}$$

Similar to (3.30) and (3.31) we can see that $\|K_2^{(1)}\| \leq C(1 + \sqrt{|h|} + \sqrt{|k|})$. We can also see that

$$|(\Psi, K_2^{(2)} \Phi)| \leq 2 \int_1^\infty t^2 dt \|\sqrt{T_p} R_{t^2}\| \|\sqrt{R_{t^2}}\| \left\| \sqrt{R_{t^2}} \frac{Y(h)}{|h|} \right\| \|R_{t^2}^{(h)}\| \|\Psi\| \|\Phi\|.$$

Then $\|K_2^{(2)}\| \leq C$ follows. $\|K_2^{(3)}\| \leq C|h|$ is similarly derived. Thus we have

$$\left\| \int_1^\infty t^2 dt R_{t^2} T_j(k+h) \frac{e^{-i(k+h)x} - e^{-ikx}}{|h|} R_{t^2} \frac{1}{\langle x \rangle^2} \right\| \leq \|K_2^{(1)}\| + \|K_2^{(2)}\| + \|K_2^{(3)}\| \leq C(1 + |k|). \quad (3.70)$$

From (3.68), (3.69) and (3.70), we obtain (3.67). ■

Lemma 3.20 *Suppose that $2|h_\mu| \leq |k_\mu|$ for $\mu = 1, 2, 3$, and $\text{supp } \hat{\varphi} \subset \{k \in \mathbb{R}^3 | |k| \leq 2\Lambda\}$ for some Λ . Then it follows that for each $k \in \mathbb{R}^3 \setminus K$,*

$$\|X_h(k)\|_{\mathcal{H}} \leq \frac{C(1 + |k|) \mathbb{1}_{|k| \leq \Lambda}}{\sqrt{\omega(k)} \sqrt{k_1^2 + k_2^2}} \|\langle x \rangle^2 \Phi_m\|_{\mathcal{H}}. \quad (3.71)$$

In particular it is satisfied that for $k \in \mathbb{R}^3 \setminus K$,

$$\lim_{h \rightarrow 0} \|X_h(k)\|_{\mathcal{H}} \leq \frac{C(1 + |k|) \mathbb{1}_{|k| \leq \Lambda}}{\sqrt{\omega(k)} \sqrt{k_1^2 + k_2^2}} \|\langle x \rangle^2 \Phi_m\|_{\mathcal{H}}. \quad (3.72)$$

Here K is defined in (2) of Proposition 3.7.

Proof: By Lemmas 3.17 and 3.18, (3.71) follows, and (3.72) is immediate from (3.71). ■

3.3.3 Weak derivative with respect to field variables

In what follows we shall see the explicit form of $\nabla_{k_\mu} \Phi_m^{(n+1)}$ by using Corollary 3.15. Let

$$\begin{aligned} X_0^\mu(k) &= (R^\mu(k)C_j(k) + R(k)C_j^\mu(k))\langle x \rangle^2 \Phi_m, \\ X_0^{\mu,n}(k) &= ((R^\mu(k)C_j(k) + R(k)C_j^\mu(k))\langle x \rangle^2 \Phi_m)^{(n)}, \quad k \in \mathbb{R}^3 \setminus K, \end{aligned}$$

where

$$\begin{aligned} R^\mu(k) &= (H_m^R - E_m + \omega(k))^{-1} \nabla_\mu \omega(k) (H_m^R - E_m + \omega(k))^{-1}, \\ C_j^\mu(k) &= \frac{4}{\pi} ((\nabla_\mu \phi_\omega(k)) I_j(k) + \phi_\omega(k) \nabla_\mu I_j(k)) + \nabla_\mu \rho_j(k) \frac{1}{\langle x \rangle^2}. \end{aligned}$$

Here $\nabla_\mu I_j(k) = \int_0^\infty dt \nabla_\mu I_j(k, t)$ and

$$\nabla_\mu I_j(k, t) = t^2 R_{t^2} T_j^\mu(k) (e^{-ikx} - 1) R_{t^2} \frac{1}{\langle x \rangle^2} + t^2 R_{t^2} T_j(k) (-ix_\mu) e^{-ikx} R_{t^2} \frac{1}{\langle x \rangle^2}$$

with $T_j^\mu(k) = \nabla_\mu e(k, j) \cdot (p - A_R)$. We fix $1 \leq \mu \leq 3$. We shall estimate

$$X_{\varepsilon_\mu}(k) - X_0^\mu(k) = \left(\frac{R(k + \varepsilon_\mu)C_j(k + \varepsilon_\mu) - R(k)C_j(k)}{\varepsilon} - R^\mu(k)C_j(k) - R(k)C_j^\mu(k) \right) \langle x \rangle^2 \Phi_m. \quad (3.73)$$

Then we divide (3.73) as

$$\begin{aligned} (3.73) &= \left(\frac{R(k + \varepsilon_\mu) - R(k)}{\varepsilon} - R^\mu(k) \right) C_j(k + \varepsilon_\mu) + R^\mu(k) (C_j(k + \varepsilon_\mu) - C_j(k)) \\ &\quad + R(k) \left(\frac{C_j(k + \varepsilon_\mu) - C_j(k)}{\varepsilon} - C_j^\mu(k) \right) \end{aligned}$$

The third term of the right-hand side is again divided as

$$\begin{aligned} \frac{C_j(k + \varepsilon_\mu) - C_j(k)}{\varepsilon} - C_j^\mu(k) &= \frac{\rho_j(k + \varepsilon_\mu) - \rho_j(k)}{\varepsilon} - \rho_j^\mu(k) \\ &\quad + \left(\frac{\phi_\omega(k + \varepsilon_\mu) - \phi_\omega(k)}{\varepsilon} - \phi_\omega^\mu(k) \right) I_j(k + \varepsilon_\mu) + \phi_\omega(k) (I_j(k + \varepsilon_\mu) - I_j(k)) \\ &\quad + \phi_\omega(k) \left(\frac{I_j(k + \varepsilon_\mu) - I_j(k)}{\varepsilon} - I_j^\mu(k) \right). \end{aligned}$$

Here $\nabla_\mu \phi_\omega(k) = \phi_\omega^\mu(k)$ and $\nabla_\mu \rho_j(k) = \rho_j^\mu(k)$. Furthermore the fourth term on the right-hand side is again divided as

$$\begin{aligned} \frac{I_j(k + \varepsilon_\mu) - I_j(k)}{\varepsilon} - I_j^\mu(k) &= \int_0^\infty dt t^2 R_{t^2} \left(\frac{T_j(k + \varepsilon_\mu) - T_j(k)}{\varepsilon} - T_j^\mu(k) \right) \xi(k + \varepsilon_\mu) R_{t^2} \frac{1}{\langle x \rangle^2} \\ &\quad + \int_0^\infty dt t^2 R_{t^2} T_j(k) (\xi(k + \varepsilon_\mu) - \xi(k)) R_{t^2} \frac{1}{\langle x \rangle^2} \\ &\quad + \int_0^\infty dt t^2 R_{t^2} T_j(k) \left(\frac{\xi(k + \varepsilon_\mu) - \xi(k)}{\varepsilon} - \xi^\mu(k) \right) R_{t^2} \frac{1}{\langle x \rangle^2}, \end{aligned}$$

where $\xi(k) = e^{-ikx} - 1$. We conclude that (3.73) can consequently be divided into the eight terms such as

$$X_{\varepsilon_\mu}(k) - X_0^\mu(k) = \sum_{j=1}^8 G_j(k),$$

where

$$\begin{aligned} G_1(k) &= \left(\frac{R(k + \varepsilon_\mu) - R(k)}{\varepsilon} - R^\mu(k) \right) C_j(k + \varepsilon_\mu) \langle x \rangle^2 \Phi_m, \\ G_2(k) &= R^\mu(k) (C_j(k + \varepsilon_\mu) - C_j(k)) \langle x \rangle^2 \Phi_m, \\ G_3(k) &= R(k) \left(\frac{\rho_j(k + \varepsilon_\mu) - \rho_j(k)}{\varepsilon} - \rho_j^\mu(k) \right) \Phi_m, \\ G_4(k) &= R(k) \left(\frac{\phi_\omega(k + \varepsilon_\mu) - \phi_\omega(k)}{\varepsilon} - \phi_\omega^\mu(k) \right) I_j(k + \varepsilon_\mu) \langle x \rangle^2 \Phi_m, \\ G_5(k) &= R(k) \phi_\omega(k) (I_j(k + \varepsilon_\mu) - I_j(k)) \langle x \rangle^2 \Phi_m, \\ G_6(k) &= R(k) \phi_\omega(k) \int_0^\infty dt t^2 R_{t^2} \left(\frac{T_j(k + \varepsilon_\mu) - T_j(k)}{\varepsilon} - T_j^\mu(k) \right) \xi(k + \varepsilon_\mu) R_{t^2} \Phi_m, \\ G_7(k) &= R(k) \phi_\omega(k) \int_0^\infty dt t^2 R_{t^2} T_j(k) (\xi(k + \varepsilon_\mu) - \xi(k)) R_{t^2} \Phi_m, \\ G_8(k) &= R(k) \phi_\omega(k) \int_0^\infty dt t^2 R_{t^2} T_j(k) \left(\frac{\xi(k + \varepsilon_\mu) - \xi(k)}{\varepsilon} - \xi^\mu(k) \right) R_{t^2} \Phi_m. \end{aligned}$$

In the definitions of $T_j^\mu(k)$ and $\rho_j^\mu(k)$, partial derivative of $e(k, j)$, $\nabla_\mu e(k, j)$, appears. The lemma below is useful to estimate $T_j^\mu(k)$ and $\rho_j^\mu(k)$.

Lemma 3.21 *There exists a constant C such that*

$$|\nabla_\mu e(k, j)| \leq \frac{C}{\sqrt{k_1^2 + k_2^2}}, \quad |\nabla_\mu^2 e(k, j)| \leq \frac{C}{k_1^2 + k_2^2}, \quad k \in \mathbb{R}^3 \setminus K, \quad \mu = 1, 2, 3.$$

Proof: The proof is straightforward. Then we show it in Appendix B. ■

We shall estimate G_1, \dots, G_8 in the following lemmas.

Lemma 3.22 *It follows that $\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^3} dk \|G_1(k)\|_{\mathcal{H}}^2 = 0$.*

Proof: We note that $|\nabla_\mu \omega(k)| = |k_\mu / \omega(k)| \leq 1$ and $|\nabla_\mu^2 \omega(k)| \leq 1 / \omega(k)$. Then

$$\begin{aligned} & \frac{R(k + \varepsilon_\mu) - R(k)}{\varepsilon} - R^\mu(k) \\ &= R(k + \varepsilon_\mu) \left(\nabla_\mu \omega(k) - \frac{\omega(k + \varepsilon_\mu) - \omega(k)}{\varepsilon} \right) R(k) + (R(k) - R(k + \varepsilon_\mu)) \nabla_\mu \omega(k) R(k). \end{aligned}$$

Hence there exists $0 \leq \theta = \theta(h, k) \leq 1$ such that

$$\begin{aligned} & \left\| \left(\frac{R(k + \varepsilon_\mu) - R(k)}{\varepsilon} - R^\mu(k) \right) \Phi \right\| \\ &= \left\| R(k + \varepsilon_\mu) \left(\frac{1}{2} \varepsilon \nabla_\mu^2 \omega(k + \theta \varepsilon_\mu) \right) R(k) \Phi \right\| + \left\| R(k) R(k + \varepsilon_\mu) \varepsilon \nabla_\mu \omega(k + \theta \varepsilon_\mu) \cdot \nabla_\mu \omega(k) R(k) \Phi \right\| \\ &\leq \frac{C|\varepsilon|}{m^3} \end{aligned}$$

and then it follows that

$$\|G_1(k)\| \leq C \frac{\varepsilon}{m^3} (|k + \varepsilon_\mu| + |k + \varepsilon_\mu|^2) \frac{\mathbb{1}_{|k| \leq \Lambda}}{\sqrt{m}} \|\langle x \rangle^2 \Phi_m\|.$$

Since the right-hand side is in $L^2(\mathbb{R}_k^3)$. Then the lemma follows. \blacksquare

Lemma 3.23 *It follows that $\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^3} dk \|G_2(k)\|^2 = 0$.*

Proof: Note that $\|C_j(k + \varepsilon_\mu) - C_j(k)\| \leq C \frac{|k| + |k|^2}{\sqrt{k_1^2 + k_2^2}} \phi_\omega(k) |\varepsilon|$. Then we can see that

$$\|G_2(k)\| \leq |\varepsilon| \frac{C}{m} \frac{|k| + |k|^2}{\sqrt{k_1^2 + k_2^2}} \frac{\mathbb{1}_{|k| \leq \Lambda}}{\sqrt{m}} \|\langle x \rangle^2 \Phi_m\|$$

and the right-hand side is in $L^2(\mathbb{R}_k^3)$. Then the lemma follows. \blacksquare

Lemma 3.24 *It follows that $\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^3} \phi(k) \|G_3(k)\| dk = 0$ for any $\phi \in C_0^\infty(\mathbb{R}^3 \setminus K)$.*

Proof: By Corollary 3.21 there exists $0 \leq \theta = \theta(h, k) \leq 1$ such that

$$\begin{aligned} & \left\| \left(\frac{\rho_j(k + \varepsilon_\mu) - \rho_j(k)}{\varepsilon} - \rho_j^\mu(k) \right) \Phi_m \right\| = |\varepsilon| \frac{1}{2} \left\| \nabla_\mu^2 \rho_j(k + \theta \varepsilon_\mu) \Phi_m \right\| \\ &= \frac{1}{2} |\varepsilon| \left\| \{ (\nabla_\mu^2 \sqrt{\omega} \hat{\varphi}) e(\cdot, j) + 2(\nabla_\mu \sqrt{\omega} \hat{\varphi}) \nabla_\mu e(\cdot, j) + \sqrt{\omega} \hat{\varphi} \nabla_\mu^2 e(\cdot, j) \} (k + \theta \varepsilon_\mu) x \Phi_m \right\| \\ &\leq |\varepsilon| C \mathbb{1}_{|k| \leq \Lambda} \left(1 + \frac{1}{\sqrt{k_1^2 + k_2^2}} + \frac{1}{k_1^2 + k_2^2} \right) \|x\| \Phi_m \end{aligned}$$

for $2|\varepsilon| \leq |k_\mu|$. Here we used $|\nabla_\mu^2 e(k + \theta \varepsilon_\mu, j)| \leq \frac{C}{k_1^2 + k_2^2}$ and $|\nabla_\mu e(k + \theta \varepsilon_\mu, j)| \leq \frac{C}{\sqrt{k_1^2 + k_2^2}}$ for $2|\varepsilon| \leq |k_\mu|$. Since $\phi \in C^\infty(\mathbb{R}^3 \setminus K)$ and then

$$\int_{\mathbb{R}^3} \phi(k) \mathbb{1}_{|k| \leq \Lambda} \left(1 + \frac{1}{\sqrt{k_1^2 + k_2^2}} + \frac{1}{k_1^2 + k_2^2} \right) dk < \infty,$$

the lemma follows. \blacksquare

Lemma 3.25 *It follows that $\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^3} \|G_4(k)\|^2 dk = 0$.*

Proof: We note that

$$|\nabla_\mu^2 \phi_\omega(k)| \leq C \mathbb{1}_{|k| \leq \Lambda} \left(\frac{1}{\sqrt{\omega}} + \frac{1}{\omega^{3/2}} + \frac{1}{\omega^{5/2}} \right) \leq C \mathbb{1}_{|k| \leq \Lambda} \left(\frac{1}{\sqrt{m}} + \frac{1}{m^{3/2}} + \frac{1}{m^{5/2}} \right)$$

and then there exists $0 \leq \theta = \theta(h, k) \leq 1$ such that

$$\begin{aligned} \left\| \left(\frac{\phi_\omega(k + \varepsilon_\mu) - \phi_\omega(k)}{\varepsilon} - \phi_\omega^\mu(k) \right) \Phi \right\| &= \frac{1}{2} |\varepsilon| \|\nabla_\mu^2 \phi_\omega(k + \theta \varepsilon_\mu) \Phi\| \\ &\leq \frac{1}{2} |\varepsilon| C \mathbb{1}_{|k| \leq \Lambda} \left(\frac{1}{\sqrt{m}} + \frac{1}{m^{3/2}} + \frac{1}{m^{5/2}} \right) \cdot \|\Phi\| \end{aligned}$$

Together with $\|\mathbb{I}_j(k + \varepsilon_\mu)\| \leq C(|k + \varepsilon_\mu| + |k + \varepsilon_\mu|^2)$ we can see that

$$\|G_4(k)\| \leq \frac{1}{2} |\varepsilon| C \mathbb{1}_{|k| \leq \Lambda} \left(\frac{1}{m} + \frac{1}{m^{3/2}} + \frac{1}{m^{5/2}} \right) (|k + \varepsilon_\mu| + |k + \varepsilon_\mu|^2) \|\langle x \rangle^2 \Phi_m\|.$$

Then the right-hand side is in $L^2(\mathbb{R}_k^3)$ and the lemma follows. ■

Lemma 3.26 *It follows that $\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^3} \phi(k) \|G_5(k)\| dk = 0$ for any $\phi \in C_0^\infty(\mathbb{R}^3 \setminus K)$.*

Proof: Since $\|\mathbb{I}_j(k + \varepsilon_\mu) - \mathbb{I}_j(k)\| \leq |\varepsilon| \frac{C(|k| + |k|^2)}{\sqrt{k_1^2 + k_2^2}}$, we can directly see that

$$\|G_5(k)\| \leq |\varepsilon| \frac{1}{m\sqrt{m}} \frac{C(|k|^2 + |k|)}{\sqrt{k_1^2 + k_2^2}} \mathbb{1}_{|k| \leq \Lambda} \|\langle x \rangle^2 \Phi_m\|.$$

Then the right-hand side is in $L^2(\mathbb{R}_k^3)$ and the lemma follows. ■

Lemma 3.27 *It follows that $\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^3} \phi(k) \|G_6(k)\| dk = 0$ for any $\phi \in C_0^\infty(\mathbb{R}^3 \setminus K)$.*

Proof: *Proof:* Let

$$G_6^{(1)} = \int_0^1 dt t^2 R_{t^2} \left(\frac{T_j(k + \varepsilon_\mu) - T_j(k)}{\varepsilon} - T_j^\mu(k) \right) \xi(k + \varepsilon_\mu) R_{t^2} \frac{1}{\langle x \rangle^2}.$$

Let $2|\varepsilon| < |k_\mu|$. Then there exists $0 \leq \theta = \theta(h, k) \leq 1$ such that

$$\begin{aligned} \left\| \left(\frac{T_j(k + \varepsilon_\mu) - T_j(k)}{\varepsilon} - T_j^\mu(k) \right) R_{t^2} \Psi \right\| &= \left\| \left(\frac{1}{2} \varepsilon T^{\mu\mu}(k + \theta \varepsilon_\mu) \right) R_{t^2} \Psi \right\| \\ &\leq \frac{C|\varepsilon|}{k_1^2 + k_2^2} \|\sqrt{T_p} R_{t^2} \Psi\|. \end{aligned}$$

Thus we obtain by (3.44) that

$$|(\Psi, G_6^{(1)} \Phi)| \leq \frac{C|\varepsilon||k|}{k_1^2 + k_2^2} \|\Psi\| \left(\int_0^1 dt t^3 \left\| |x| R_{t^2} \frac{1}{\langle x \rangle^2} \Phi \right\|^2 \right)^{\frac{1}{2}} \leq \frac{C|\varepsilon||k|}{k_1^2 + k_2^2} \|\Psi\| \|\Phi\|.$$

Let

$$\begin{aligned} G_6^{(21)} &= \int_1^\infty dt t^2 R_{t^2} \left(\frac{T_j(k + \varepsilon_\mu) - T_j(k)}{\varepsilon} - T_j^\mu(k) \right) R_{t^2}^{(k + \varepsilon_\mu)} \frac{\xi(k + \varepsilon_\mu)}{\langle x \rangle^2}, \\ G_6^{(22)} &= -2 \int_1^\infty dt t^2 R_{t^2} \left(\frac{T_j(k + \varepsilon_\mu) - T_j(k)}{\varepsilon} - T_j^\mu(k) \right) R_{t^2} Y(k + \varepsilon_\mu) R_{t^2}^{(k + \varepsilon_\mu)} \frac{1}{\langle x \rangle^2}, \\ G_6^{(23)} &= -|k + \varepsilon_\mu|^2 \int_1^\infty dt t^2 R_{t^2} \left(\frac{T_j(k + \varepsilon_\mu) - T_j(k)}{\varepsilon} - T_j^\mu(k) \right) R_{t^2} R_{t^2}^{(k + \varepsilon_\mu)} \frac{1}{\langle x \rangle^2}. \end{aligned}$$

Then

$$\int_1^\infty dt t^2 R_{t^2} \left(\frac{T_j(k + \varepsilon_\mu) - T_j(k)}{\varepsilon} - T_j^\mu(k) \right) \xi(k + \varepsilon_\mu) R_{t^2} \frac{1}{\langle x \rangle^2} = G_6^{(21)} + G_6^{(22)} + G_6^{(23)}.$$

Let $2|\varepsilon| < |k_\mu|$. Then note also that there exists $0 \leq \theta = \theta(h, k) \leq 1$ such that

$$\begin{aligned} \left| \left(\Psi, \left(\frac{T_j(k + \varepsilon_\mu) - T_j(k)}{\varepsilon} - T_j^\mu(k) \right) \Phi \right) \right| &= \left| \left(\Psi, \left(\frac{1}{2} \varepsilon T_j^{\mu\mu}(k + \theta \varepsilon_\mu) \right) \Phi \right) \right| \\ &\leq \frac{C|\varepsilon|}{k_1^2 + k_2^2} \|T_p^{\frac{1}{4}} \Psi\| \|T_p^{\frac{1}{4}} \Phi\|. \end{aligned}$$

Then in a similar manner to (3.35) it can be estimated as

$$\begin{aligned} &\left\| \int_1^\infty dt t^2 R_{t^2} \left(\frac{T_j(k + \varepsilon_\mu) - T_j(k)}{\varepsilon} - T_j^\mu(k) \right) \xi(k + \varepsilon_\mu) R_{t^2} \frac{1}{\langle x \rangle^2} \right\| \\ &\leq \|G_6^{(21)}\| + \|G_6^{(22)}\| + \|G_6^{(23)}\| \leq \frac{C|\varepsilon|(|k| + |k|^2)}{k_1^2 + k_2^2} \end{aligned}$$

and we have

$$\|G_6(k)\| \leq \frac{C|\varepsilon|(|k| + |k|^2)}{k_1^2 + k_2^2} \frac{1}{m\sqrt{m}} \mathbb{1}_{|k| \leq \Lambda} \|\langle x \rangle^2 \Phi_m\|$$

for $2|\varepsilon| < |k_\mu|$. Thus the lemma follows. ■

Lemma 3.28 *It follows that $\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^3} \|G_7(k)\|^2 dk = 0$.*

Proof: Let

$$G_7^{(1)} = \int_0^1 dt t^2 R_{t^2} T_j(k) e^{-ikx} (e^{-i\varepsilon_\mu x} - 1) R_{t^2} \frac{1}{\langle x \rangle^2}.$$

Then we have

$$|(\Psi, G_7^{(1)}\Phi)| \leq C|\varepsilon| \|\Psi\| \left(\int_0^1 dt t^3 \left\| |x| R_{t^2} \frac{1}{\langle x \rangle^2} \Phi \right\|^2 \right)^{\frac{1}{2}} \leq C|\varepsilon| \|\Psi\| \|\Phi\|.$$

We set

$$\begin{aligned} G_7^{(21)} &= \int_1^\infty dt t^2 R_{t^2} T_j(k) e^{-ikx} R_{t^2}^{(\varepsilon_\mu)} (e^{-i\varepsilon_\mu x} - 1) \frac{1}{\langle x \rangle^2}, \\ G_7^{(21)} &= -2 \int_1^\infty dt t^2 R_{t^2} T_j(k) e^{-ikx} R_{t^2} Y(h) R_{t^2}^{(\varepsilon_\mu)} \frac{1}{\langle x \rangle^2}, \\ G_7^{(23)} &= -|\varepsilon|^2 \int_1^\infty dt t^2 R_{t^2} T_j(k) e^{-ikx} R_{t^2} R_{t^2}^{(\varepsilon_\mu)} \frac{1}{\langle x \rangle^2}. \end{aligned}$$

Then

$$\int_1^\infty dt t^2 R_{t^2} T_j(k) (\xi(k + \varepsilon_\mu) - \xi(k)) R_{t^2} \frac{1}{\langle x \rangle^2} = G_7^{(21)} + G_7^{(22)} + G_7^{(23)}.$$

For all $\Psi, \Phi \in \mathcal{H}$, we have

$$|(\Psi, G_7^{(21)}\Phi)| \leq \int_1^\infty dt t^2 \|T_p^{\frac{1}{4}} R_{t^2} \Psi\| \|T_p^{\frac{1}{4}} e^{-ikx} R_{t^2}^{(\varepsilon_\mu)} \tilde{\Phi}\|,$$

where $\tilde{\Phi} = (e^{-i\varepsilon_\mu x} - 1) \frac{1}{\langle x \rangle^2} \Phi$. Note that

$$\begin{aligned} \|T_p^{\frac{1}{4}} e^{-ikx} R_{t^2}^{(\varepsilon_\mu)} \tilde{\Phi}\| &= \|e^{-ikx} |(p + \varepsilon_\mu - A_R - (\varepsilon_\mu + k))^2|^{\frac{1}{4}} R_{t^2}^{(\varepsilon_\mu)} \tilde{\Phi}\| \\ &\leq \|T_{p+\varepsilon_\mu}^{\frac{1}{4}} R_{t^2}^{(\varepsilon_\mu)} \tilde{\Phi}\| + (\sqrt{|\varepsilon|} + \sqrt{|k|}) \|R_{t^2}^{(\varepsilon_\mu)} \tilde{\Phi}\|. \end{aligned}$$

Similar to (3.30) and (3.31) we can see that $\|G_7^{(21)}\| \leq C(1 + \sqrt{|\varepsilon|} + \sqrt{|k|})$. We can also see that

$$|(\Psi, G_7^{(22)}\Phi)| \leq 2 \int_1^\infty dt t^2 \|\sqrt{T_p} R_{t^2}\| \|\sqrt{R_{t^2}}\| \|\sqrt{R_{t^2}} Y(h)\| \|R_{t^2}^{(\varepsilon_\mu)}\| \|\Psi\| \|\Phi\|.$$

Then $\|G_7^{(22)}\| \leq |\varepsilon|C$ follows. $\|G_7^{(23)}\| \leq |\varepsilon|C$ is similarly derived. Thus we have

$$\left\| \int_1^\infty dt t^2 R_{t^2} T_j(k) (\xi(k + \varepsilon_\mu) - \xi(k)) R_{t^2} \frac{1}{\langle x \rangle^2} \right\| \leq \|G_7^{(21)}\| + \|G_7^{(22)}\| + \|G_7^{(23)}\| \leq |\varepsilon|C(1 + |k|).$$

Then

$$\|G_7(k)\| \leq \frac{|\varepsilon|C(1 + |k|)}{m\sqrt{m}} \mathbb{1}_{|k| \leq \Lambda} \|\langle x \rangle^2 \Phi_m\|$$

and the right-hand side is in $L^2(\mathbb{R}_k^3)$. Then the lemma follows. ■

Lemma 3.29 *It follows that $\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^3} \|G_8(k)\|^2 dk = 0$.*

Proof: Let

$$G_8^{(1)} = \int_0^1 dt t^2 R_{t^2} T_j(k + \varepsilon_\mu) e^{-ikx} \left(\frac{e^{-i\varepsilon_\mu x} - 1}{\varepsilon} + ix_\mu \right) R_{t^2} \frac{1}{\langle x \rangle^2}.$$

We also obtain that

$$|(\Psi, G_8^{(1)} \Phi)| \leq C|\varepsilon| \|\Psi\| \left(\int_0^1 dt t^3 \left\| |x|^2 R_{t^2} \frac{1}{\langle x \rangle^2} \Phi \right\|^2 \right)^{\frac{1}{2}} \leq C \|\Psi\| \|\Phi\|.$$

Here we used (3.37). Next we set

$$\begin{aligned} G_8^{(21)} &= \int_1^\infty dt t^2 R_{t^2} T_j(k) e^{-ikx} R_{t^2}^{(\varepsilon_\mu)} \left(\frac{e^{-i\varepsilon_\mu x} - 1}{\varepsilon} + ix_\mu \right) \frac{1}{\langle x \rangle^2}, \\ G_8^{(22)} &= \int_1^\infty dt t^2 R_{t^2} T_j(k) e^{-ikx} R_{t^2} \left(-2 \frac{Y(\varepsilon_\mu)}{\varepsilon} - \varepsilon \right) R_{t^2}^{(\varepsilon_\mu)} \varepsilon \nabla_\mu \omega(k + \theta \varepsilon_\mu) R_{t^2} \frac{1}{\langle x \rangle^2}, \\ G_8^{(23)} &= \int_1^\infty dt t^2 R_{t^2} T_j(k) e^{-ikx} R_{t^2}^{(\varepsilon_\mu)} R_{t^2} \varepsilon \nabla_\mu \omega(k + \theta \varepsilon_\mu) ix_\mu \frac{1}{\langle x \rangle^2}, \\ G_8^{(24)} &= -\varepsilon^2 \int_1^\infty dt t^2 R_{t^2} T_j(k) e^{-ikx} R_{t^2}^{(\varepsilon_\mu)} R_{t^2} \frac{1}{\langle x \rangle^2}. \end{aligned}$$

Then

$$\int_1^\infty dt t^2 R_{t^2} T_j(k) e^{-ikx} \left(\frac{e^{-i\varepsilon_\mu x} - 1}{\varepsilon} + ix_\mu \right) R_{t^2} \frac{1}{\langle x \rangle^2} = G_8^{(21)} + G_8^{(22)} + G_8^{(23)} + G_8^{(24)}.$$

For all $\Psi, \Phi \in \mathcal{H}$, we have

$$|(\Psi, G_8^{(21)} \Phi)| \leq \int_1^\infty dt t^2 \|T_p^{\frac{1}{4}} R_{t^2} \Psi\| \|T_p^{\frac{1}{4}} e^{-ikx} R_{t^2}^{(\varepsilon_\mu)} \tilde{\Phi}\|,$$

where $\tilde{\Phi} = \left(\frac{e^{-i\varepsilon_\mu x} - 1}{\varepsilon} + ix_\mu \right) \frac{1}{\langle x \rangle^2} \Phi$. Then we have $\|G_8^{(21)}\| \leq C(1 + \sqrt{|\varepsilon|} + \sqrt{|k|})$. We can also see that

$$|(\Psi, G_8^{(22)} \Phi)| \leq |\varepsilon| \int_1^\infty dt t^2 \|\sqrt{T_p} R_{t^2}\| \|\sqrt{R_{t^2}}\| \left\| \sqrt{R_{t^2}} \left(\frac{Y(\varepsilon)}{\varepsilon} + \varepsilon \right) \right\| \|R_{t^2}^{(\varepsilon_\mu)} R_{t^2}\| \|\Psi\| \|\Phi\|.$$

Then $\|G_8^{(22)}\| \leq C|\varepsilon|$ follows. $\|G_8^{(23)}\| \leq C|\varepsilon|$ and $\|G_8^{(24)}\| \leq C|\varepsilon|$ are similarly derived. Thus we have

$$\left\| \int_1^\infty dt t^2 R_{t^2} T_j(k) \frac{\xi(k + \varepsilon_\mu) - \xi(k)}{\varepsilon} R_{t^2} \frac{1}{\langle x \rangle^2} \right\| \leq \|G_8^{(21)}\| + \|G_8^{(22)}\| + \|G_8^{(23)}\| \leq |\varepsilon| C(1 + |k|).$$

Hence

$$\|G_8(k)\| \leq |\varepsilon| \frac{C(1+|k|)}{m\sqrt{m}} \mathbb{1}_{|k| \leq \Lambda} \|\langle x \rangle^2 \Phi_m\|.$$

The right-hand side is in $L^2(\mathbb{R}_k^3)$ and the proof is completed. \blacksquare

We define X_0^μ and $X_0^{\mu,n+1}$ by the constant fiber direct integral of $X_0^\mu(k)$ and $X_0^{\mu,n}(k)$. Let

$$\begin{aligned} X_0^\mu &= \int_{\mathbb{R}^3}^\oplus X_0^\mu(k) dk \in \int_{\mathbb{R}^3}^\oplus dk \mathcal{H}, \\ X_0^{\mu,n+1} &= \int_{\mathbb{R}^3}^\oplus X_0^{\mu,n}(k) dk \in \int_{\mathbb{R}^3}^\oplus dk \mathcal{H}^{(n)}. \end{aligned}$$

Lemma 3.30 *Let $f \in C_0^\infty(\mathbb{R}^3)$ and $G \in \mathcal{H}^{(n)}$. Then it follows that*

$$\lim_{\varepsilon \rightarrow 0} (f \otimes G, X_{\varepsilon_\mu}^{n+1}) = (f \otimes G, X_0^{\mu,n+1}).$$

In particular it follows that

$$(\nabla_\mu f \otimes G, \Phi_m^{(n+1)}) = \frac{1}{\sqrt{n+1}} (f \otimes G, X_0^{\mu,n+1}).$$

Proof: We have $(\nabla_\mu f \otimes G, \Phi_m^{(n+1)}) = \lim_{j \rightarrow \infty} (\nabla_\mu f_j \otimes G, \Phi_m^{(n+1)})$, where $f_j \in C_0^\infty(\mathbb{R}^3 \setminus K)$ and $\nabla_\mu f_j \rightarrow \nabla_\mu f$ in $L^2(\mathbb{R}_k^3)$. Note that $X_{\varepsilon_\mu}(k) - X_0^\mu(k) = \sum_{j=1}^8 G_j(k)$ and then

$$\left| \int_{\mathbb{R}^3} f_j(k) (G, X_{\varepsilon_\mu}^n(k) - X_0^{\mu,n}(k))_{\mathcal{H}^{(n)}} dk \right| \leq \|G\| \sum_{j=1}^8 \int_{\mathbb{R}^3} |f_j(k)| \|G_j(k)\|_{\mathcal{H}} dk \rightarrow 0$$

as $\varepsilon \rightarrow 0$ by Lemmas 3.22–3.29. Then

$$\begin{aligned} (\nabla_\mu f \otimes G, \Phi_m^{(n+1)})_{\mathcal{H}^{(n+1)}} &= \lim_{j \rightarrow \infty} (\nabla_\mu f_j \otimes G, \Phi_m^{(n+1)})_{\mathcal{H}^{(n+1)}} = \lim_{j \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} (f_j \otimes G, X_{\varepsilon_\mu}^{n+1})_{\mathcal{H}^{(n+1)}} \\ &= \lim_{j \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^3} f_j(k) (G, X_{\varepsilon_\mu}^n(k))_{\mathcal{H}^{(n)}} dk = \lim_{j \rightarrow \infty} \int_{\mathbb{R}^3} f_j(k) (G, X_0^{\mu,n}(k))_{\mathcal{H}^{(n)}} dk = (f \otimes G, X_0^{\mu,n+1}) \end{aligned}$$

follows. Then the proof is complete. \blacksquare

Lemma 3.31 *For arbitrary $n \geq 1$, $\Phi_m^{(n)}$ is weakly differentiable with respect to $k_{j,\mu}$, $j = 1, \dots, n, \mu = 1, 2, 3$. Moreover if $1 \leq p < 2$ and $\Omega \subset \mathbb{R}_x^3 \times \mathbb{R}_k^{3n}$ is bounded, then*

$$\sup_{0 < m < m_0} \|\nabla_{k_{i,\mu}} \Phi_m^{(n)}\|_{L^p(\Omega)} < \infty. \quad (3.74)$$

Proof: Let $n \geq 0$. By Lemma 3.30 we can conclude that

$$\nabla_\mu \Phi_m^{(n+1)} = \frac{1}{\sqrt{n+1}} X_0^{\mu, n+1}.$$

We then see that

$$\|\nabla_\mu \Phi_m^{(n+1)}\|_{L^p(\Omega)}^p \leq C \int_{\mathbb{R}^3} dk \|X_0^{n, \mu}(k)\|_{\mathcal{H}^{(n)}}^p.$$

By (3.72) we have

$$\int_{\mathbb{R}^3} dk \|X_0^{n, \mu}(k)\|_{\mathcal{H}^{(n)}}^p \leq \int_{\mathbb{R}^3} dk \left| \frac{(1+|k|)\mathbb{1}_{|k| \leq \Lambda}(k)}{\sqrt{\omega(k)}\sqrt{k_1^2 + k_2^2}} \right|^p \|\langle x \rangle^2 \Phi_m\|^p < \infty$$

for $p < 2$. Furthermore we can see that

$$\int_{\mathbb{R}^3} dk \left| \frac{(1+|k|)\mathbb{1}_{|k| \leq \Lambda}(k)}{\sqrt{\omega(k)}\sqrt{k_1^2 + k_2^2}} \right|^p \|\langle x \rangle^2 \Phi_m\|^p \leq C \int_{\mathbb{R}^3} dk \left| \frac{(1+|k|)\mathbb{1}_{|k| \leq \Lambda}(k)}{\sqrt{|k|}\sqrt{k_1^2 + k_2^2}} \right|^p < \infty,$$

where $C = \sup_m \|\langle x \rangle^2 \Phi_m\|^2 < \infty$. Thus (3.74) follows. \blacksquare

3.3.4 Weak derivative with respect to particle variables

We consider the weak derivative of $\Phi_m^{(n)}(x, k_1, \dots, k_n)$ with respect to x .

Lemma 3.32 *For arbitrary $n \geq 0$, $\Phi_m^{(n)}$ is weakly differentiable with respect to x_μ , $\mu = 1, 2, 3$. Moreover if $1 \leq p < 2$ and $R > 0$. Then*

$$\sup_{0 < m < m_0} \|\mathbb{1}_{|x| \leq R} \nabla_{x_\mu} \Phi_m^{(n)}\|_{L^p(\mathbb{R}_x^3 \times \mathbb{R}_k^{3n})} < \infty. \quad (3.75)$$

Proof: Note that $|p|$ is relatively bounded with respect to H_m^R . We have

$$\|\nabla_{x_\mu} \Phi_m\|_{\mathcal{H}} \leq \| |p| \Phi_m \|_{\mathcal{H}} \leq C \|(H_m^R + \mathbb{1}) \Phi_m\|_{\mathcal{H}}. \quad (3.76)$$

Since $H_m^R \leq H_{m_0}^R$ it holds that $E_m \leq E_{m_0}$. Thus $\|\nabla_{x_\mu} \Phi_m\|_{\mathcal{H}} \leq C(E_{m_0} + 1)$. In particular $\|\mathbb{1}_{|x| \leq R} \nabla_{x_\mu} \Phi_m^{(n+1)}\|_{\mathcal{H}^{(n+1)}} \leq C(E_{m_0} + 1)$. We then have

$$\|\mathbb{1}_{|x| \leq R} \nabla_{x_\mu} \Phi_m\|_{L^p(\mathbb{R}_x^3 \times \mathbb{R}_k^{3n})}^p \leq C \|\nabla_{x_\mu} \Phi_m\|_{\mathcal{H}^{(n+1)}}^p \leq C(E_{m_0} + 1)^p.$$

Then the lemma follows. \blacksquare

3.4 Existence of ground states

Now we state the main theorem.

Theorem 3.33 *Let $m = 0$. Suppose that $\int_{\mathbb{R}^3} \left(\frac{|k|+|k|^2}{\omega(k)} \right)^2 \frac{\hat{\varphi}(k)^2}{\omega(k)} dk < \infty$. Then H_0^R has the ground state and it is unique up to multiple constant. In particular H_m for $m = 0$ has the ground state.*

Proof: The uniqueness is shown in [Hir14]. There exists a sequence of normalized ground states $\{\Phi_{m_j}\}_{j=1}^\infty$ such that $\lim_{j \rightarrow \infty} m_j = 0$ and $w\text{-}\lim \Phi_{m_j} = \Phi$. We see that $H_0^R \Phi = E\Phi$ in the same way as [GLL01, Step 1 of the proof of Theorem 2.1]. Hence it is enough to show that $\Phi \neq 0$. Let ϵ be an arbitrary positive number. Take a sufficient large number $R > 0$, and let $B_R = \{X \in \mathbb{R}_x^3 \times \mathbb{R}_k^{3n} \mid |X|^2 < R\}$ be the ball with radius R centered at the origin. Take also a sufficient large M and R . Then

$$\begin{aligned} \|\Phi_{m_j} - \Phi_{m_k}\|_{\mathcal{H}}^2 &\leq \sum_{n=0}^M \|\Phi_{m_j}^{(n)} - \Phi_{m_k}^{(n)}\|_{\mathcal{H}^{(n)}}^2 + \sum_{n=M+1}^\infty \|\Phi_{m_j}^{(n)} - \Phi_{m_k}^{(n)}\|_{\mathcal{H}^{(n)}}^2 \\ &\leq \sum_{n=0}^M \|\Phi_{m_j}^{(n)} - \Phi_{m_k}^{(n)}\|_{L^2(B_R)}^2 + \frac{\sup_j \|(1+|x|)\Phi_{m_j}\|_{L^2(B_R^c)}^2}{1+R} + \frac{2}{M} \sup_j \|N^{\frac{1}{2}}\Phi_{m_j}\|_{\mathcal{H}}^2. \end{aligned}$$

$\sup_j \|(1+|x|)\Phi_{m_j}\|_{L^2(B_R^c)}^2 < C_1$ and $\sup_j \|N^{\frac{1}{2}}\Phi_{m_j}\|_{\mathcal{H}}^2 < C_2$ are derived from spatial decay (2.6) and bound (3.47), respectively. Hence

$$\|\Phi_{m_j} - \Phi_{m_k}\|_{\mathcal{H}}^2 < \sum_{n=0}^M \|\Phi_{m_j}^{(n)} - \Phi_{m_k}^{(n)}\|_{L^2(B_R)}^2 + \frac{\epsilon}{2}.$$

By Lemmas 3.31 and 3.32, $\{\|\Phi_{m_j}^{(n)}\|_{W^{1,p}(B_R)}\}_{j=1}^\infty$ is bounded for each $p \in (1, 2)$. Thus we see that $w\text{-}\lim_{j \rightarrow \infty} \Phi_{m_j}^{(n)} = \Phi^{(n)}$ in $W^{1,p}(B_R)$. We can apply the Rellich-Kondrachov theorem, and see that $\Phi_{m_j}^{(n)}$ strongly converges to $\Phi^{(n)}$ in $L^q(B_R)$ with $1 \leq q < \frac{12p}{12-p}$. In particular, taking $p > 12/7$, we have for all $n \geq 0$, $s\text{-}\lim_{j \rightarrow \infty} \Phi_{m_j}^{(n)} = \Phi^{(n)}$ in $L^2(B_R)$. Thus $\{\Phi_{m_j}\}$ is Cauchy and we can see that $s\text{-}\lim_j \Phi_{m_j} = \Phi$. Hence $\Phi \neq 0$ and the proof is complete. ■

We immediately have the corollary below:

Corollary 3.34 *Let $m = 0$. Suppose that*

$$\int_{\mathbb{R}^3} \left(\frac{1}{|k|} + 1 + |k| \right) \hat{\varphi}(k)^2 dk < \infty.$$

Let us introduce a coupling constant α as

$$H_\alpha = |p - \alpha A| + V + H_{f,0}.$$

Then H_α is self-adjoint on $D(|p|) \cap D(H_{f,0})$ and has the ground state for arbitrary $\alpha \in \mathbb{R}$.

A Asymptotic field and number operator

We quickly reviews a Hilbert space-valued integral operator which is the so-called Carleman operator for the self-consistency of the paper. See [Hir05] for explicit statement and conditions. Let

$$a_t(f, j) = e^{-itH_m^R} e^{itH_{f,m}} a(f, j) e^{-itH_{f,m}} e^{itH_m^R}.$$

Since

$$a_t(f, j)\Phi_m = e^{-it(H_m^R - E_m)} a(e^{-it\omega} f, j)\Phi_m,$$

we can see that

$$\text{s-}\lim_{t \rightarrow \pm\infty} a_t(f, j)\Phi_m = 0$$

by the Riemman-Lebesgue lemma, and by which and the identity

$$(\Phi, a_t(f, j)\Phi_m) - (\Phi, a(f, j)\Phi_m) = \int_0^t \frac{d}{ds} (\Phi, a_s(f, j)\Phi_m) ds,$$

we can also see that

$$\begin{aligned} 0 &= \lim_{t \rightarrow \pm\infty} (\Phi, a_t(f, j)\Phi_m) \\ &= (\Phi, a(f, j)\Phi_m) + i \int_{\mathbb{R}^3} dk \int_0^\infty ds (\Phi, f(k) e^{-is(H_m^R - E_m + \omega(k))} C_j(k) \langle x \rangle^2 \Phi_m). \end{aligned}$$

Then we have

$$a(f, j)\Phi_m = - \int_0^\infty f(k) \kappa_j(k) dk.$$

Here $\kappa_j(k) = (H_m^R - E_m + \omega(k))^{-1} C_j \langle x \rangle^2$. Let $T_{gj} : L^2(\mathbb{R}^3) \rightarrow \mathcal{H}$ be defined by

$$T_{gj} f = - \int_{\mathbb{R}^3} f(k) \kappa_j(k) dk.$$

Here T_{gj} is a \mathcal{H} -valued integral operator, and $a(f, j)\Phi_m = -T_{gj} f$. Adjoint

$$T_{gj}^* : \mathcal{H} \ni \Phi \mapsto -(\kappa_j(\cdot), \Phi)_{\mathcal{H}} \in L^2(\mathbb{R}^3)$$

is called a Carleman operator [Wei80, p.141] with

$$\text{D}(T_{gj}^*) = \{\Phi \in \mathcal{F} | (\kappa_j(\cdot), \Phi)_{\mathcal{H}} \in L^2(\mathbb{R}^3)\}.$$

Thus it is known [Wei80, Theorem 6.12] that T_{gj}^* is a Hilbert-Schmidt if and only if $\|\kappa_j(\cdot)\|_{\mathcal{H}}$ is L^2 , i.e.,

$$\|T_{gj}^*\|_{\text{HS}}^2 = \int_{\mathbb{R}^3} \|\kappa_j(k)\|_{\mathcal{H}}^2 dk < \infty,$$

which implies that $\int_{\mathbb{R}^3} \|\kappa_j(k)\|^2 dk < \infty$ if and only if T_{gj} is Hilbert-Schmidt, and when T_{gj} is Hilbert-Schmidt we can see that

$$\|T_{gj}\|_{\text{HS}}^2 = \|T_{gj}^*\|_{\text{HS}}^2 = \int_{\mathbb{R}^3} \|\kappa_j(k)\|_{\mathcal{H}}^2 dk.$$

Furthermore by the definition of the Hilbert-Schmidt norm we see that

$$\|T_{gj}\|_{\text{HS}}^2 = \sum_{j=1,2} \sum_{M=1}^{\infty} \|T_{gj}e_m\|_{\mathcal{H}}^2$$

for any complete orthonormal system $\{e_m\}_{m=1}^{\infty}$ in $L^2(\mathbb{R}^3)$. Thus it is noticed that

$$\sum_{j=1,2} \sum_{m=1}^{\infty} \|T_{gj}e_m\|_{\mathcal{H}}^2 = \sum_{j=1,2} \sum_{m=1}^{\infty} \|a(e_m, j)\Phi_m\|_{\mathcal{H}}^2 = \|N^{\frac{1}{2}}\Phi_m\|_{\mathcal{H}}^2.$$

The second identity is straightforwardly derived. Hence we can conclude that $\Phi_m \in D(N^{\frac{1}{2}})$ if and only if $\int_{\mathbb{R}^3} \|\kappa_j(k)\|_{\mathcal{H}}^2 dk < \infty$ and when $\Phi_m \in D(N^{\frac{1}{2}})$, it follows that

$$\|N^{\frac{1}{2}}\Phi_m\|^2 = \int_{\mathbb{R}^3} \|\kappa_j(k)\|_{\mathcal{H}}^2 dk.$$

B Derivative of polarization vectors

We give a proof of Lemma 3.21.

Proof: Let us set $X = \sqrt{k_1^2 + k_2^2}$. Then

$$e(k, 1) = \left(\frac{k_2}{X}, \frac{-k_1}{X}, 0\right), e(k, 2) = \left(\frac{k_1 k_3}{|k|X}, \frac{-k_2 k_3}{|k|X}, \frac{-X}{|k|}\right), e(k, 3) = \left(\frac{k_1}{|k|}, \frac{k_2}{|k|}, \frac{k_3}{|k|}\right).$$

Using formulas $\nabla_\mu |k| = k_\mu/|k|$ for $\mu = 1, 2, 3$, and $\nabla_\mu X = k_\mu/X$ for $\mu = 1, 2$, we can see that

$$\begin{aligned}
e^1(k, 1) &= \left(-\frac{k_1 k_2}{X^3}, \frac{-k_2^2}{X^3}, 0\right), \\
e^1(k, 2) &= \left(\frac{k_3(X^2|k|^2 - k_1^2(X^2 + |k|^2))}{X^3|k|^3}, k_1 k_2 k_3 \frac{X^2 + |k|^2}{X^3|k|^3}, \frac{-k_1 X^2 + |k|^2 k_1}{X|k|^3}\right), \\
e^1(k, 3) &= \left(\frac{k_2^2 + k_3^2}{|k|^3}, \frac{-k_1 k_2}{|k|^3}, \frac{-k_1 k_3}{|k|^3}\right), \\
e^2(k, 1) &= \left(\frac{k_1^2}{X^3}, \frac{-k_1 k_2}{X^3}, 0\right), \\
e^2(k, 2) &= \left(k_1 k_2 k_3 \frac{-X^2 - |k|^2}{X^3|k|^3}, \frac{-k_3(X^2|k|^2 - k_2^2(X^2 + |k|^2))}{X^3|k|^3}, \frac{-k_2 X^2 + |k|^2 k_2}{X|k|^3}\right), \\
e^2(k, 3) &= \left(\frac{-k_1 k_2}{|k|^3}, \frac{k_1^2 + k_3^2}{|k|^3}, \frac{-k_2 k_3}{|k|^3}\right), \\
e^3(k, 1) &= \left(\frac{-k_2 k_3}{X^3}, \frac{-k_1 k_3}{X^3}, 0\right), \\
e^3(k, 2) &= \left(k_1 \frac{X^2|k|^2 - k_3^2(X^2 + |k|^2)}{X^3|k|^3}, -k_2 \frac{X^2|k|^2 - k_3^2(X^2 + |k|^2)}{X^3|k|^3}, \frac{-k_3^3}{X|k|^3}\right), \\
e^3(k, 3) &= \left(\frac{-k_1 k_3}{|k|^3}, \frac{-k_2 k_3}{|k|^3}, \frac{k_1^2 + k_2^2}{|k|^3}\right),
\end{aligned}$$

and

$$\begin{aligned}
e^{11}(k, 1) &= \left(\frac{-k_2(X^2 - 3k_1^2)}{X^5}, \frac{3k_2^2 k_1}{X^5}, 0\right), \\
e^{11}(k, 3) &= \left(\frac{-3k_1(k_2^2 + k_3^2)}{|k|^5}, \frac{-k_2(|k|^2 - 3k_1^2)}{|k|^5}, \frac{-k_3(|k|^2 - 3k_1^2)}{|k|^5}\right), \\
e^{22}(k, 1) &= \left(\frac{-3k_1^2 k_2}{X^5}, \frac{-k_1(X^2 - 3k_2^2)}{X^5}, 0\right), \\
e^{22}(k, 3) &= \left(\frac{-k_1(|k|^2 - 3k_2^2)}{|k|^5}, \frac{-3k_2(k_1^2 + k_3^2)}{|k|^5}, \frac{-k_3(|k|^2 - 3k_2^2)}{|k|^5}\right), \\
e^{33}(k, 1) &= \left(\frac{-k_2(X^2 - 3k_3^2)}{X^5}, \frac{-k_1(X^2 - 3k_3^2)}{X^5}, 0\right), \\
e^{33}(k, 3) &= \left(\frac{-k_1(|k|^2 - 3k_3^2)}{|k|^5}, \frac{-k_2(|k|^2 - 3k_3^2)}{|k|^5}, \frac{-3k_3(k_1^2 + k_2^2)}{|k|^5}\right).
\end{aligned}$$

Finally $e^{jj}(k, 2) = (A_j, B_j, C_j)$, $j = 1, 2, 3$, with

$$\begin{aligned}
A_1 &= -k_1 k_3 \left\{ \frac{X^2 + |k|^2}{X^3 |k|^3} + \left(\frac{2X^2 |k|^2 - k_1^2 (|k|^2 + 3X^2)}{X^3 |k|^5} + \frac{2X^2 |k|^2 - k_1^2 (3|k|^2 - X^2)}{X^5 |k|^3} \right) \right\}, \\
B_1 &= k_2 k_3 \left(\frac{X^2 |k|^2 - k_1^2 (3X^2 + |k|^2)}{X^3 |k|^5} + \frac{X^2 |k|^2 - k_1^2 (3|k|^2 + X)}{X^5 |k|^3} \right), \\
C_1 &= -\frac{X^2 |k|^2 - k_1^2 (X^2 + |k|^2)}{X^3 |k|^3} + \frac{(X^2 + k_1^2) |k|^2 - 3X^2 k_1^2}{X |k|^5}, \\
A_2 &= -k_1 k_3 \left(\frac{X^2 |k|^2 - k_2^2 (3X^2 + |k|^2)}{X^3 |k|^5} + \frac{X^2 |k|^2 - k_2^2 (X^2 + 3|k|^2)}{X^5 |k|^3} \right), \\
B_2 &= k_2 k_3 \left\{ \frac{X^2 + |k|^2}{X^3 |k|^3} + \left(\frac{2X^2 |k|^2 - k_2^2 (3X^2 + |k|^2)}{X^3 |k|^5} + \frac{2X^2 |k|^2 - k_2^2 (X^2 + 3|k|^2)}{X^5 |k|^3} \right) \right\}, \\
C_2 &= -\frac{X^2 |k|^2 - k_2^2 (X^2 + |k|^2)}{X^3 |k|^3} + \frac{(k_2^2 + X^2) |k|^2 - 3k_2^2 X^2}{X |k|^5}, \\
A_3 &= -k_1 k_3 \left\{ \frac{X^2 + |k|^2}{X^3 |k|^3} + \left(\frac{2X^2 |k|^2 - k_3^2 (3X^2 + |k|^2)}{X^3 |k|^5} + \frac{2X^2 |k|^2 - k_3^2 (X^2 + 3|k|^2)}{X^5 |k|^3} \right) \right\}, \\
B_3 &= k_2 k_3 \left\{ \frac{X^2 + |k|^2}{X^3 |k|^3} + \left(\frac{2X^2 |k|^2 - k_3^2 (3X^2 + |k|^2)}{X^3 |k|^5} + \frac{2X^2 |k|^2 - k_3^2 (X^2 + 3|k|^2)}{X^5 |k|^3} \right) \right\}, \\
C_3 &= \frac{3X^2 |k|^2 - k_3^2 (|k|^2 + 3X^2)}{X^3 |k|^5} k_3^2.
\end{aligned}$$

Then the lemma is proven. ■

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